A study on transitive modal logics

Serenella Cerrito[1] and Marta Cialdea Mayer[2*]

[1] Université de Paris-Sud, LRI, Bât 490,
F-91405 Orsay Cedex, France.
e-mail: serena@lri.fr

[2] Dipartimento di Informatica e Automazione, Terza Università di Roma,
via della Vasca Navale 84, 00146 Roma, Italia.
e-mail: cialdea@inf.uniroma3.it

Abstract. The concern of this paper is the study of automated deduction methods for propositional modal logics. We use tableau proof-systems to show that Fitting's translation of the transitive modal logic **S4** into **T** can be constructed in deterministic polynomial time. Moreover, we establish a polynomial bound to the length of branches in both tableau and sequent proof search for the transitive logics **S4** and **K4**. This allows the elimination of periodicity tests when proving **S4**-validity; moreover, it provides directly a form of "contraction elimination result" in modal sequent calculi, in the sense that the number of contractions needed in a branch of a sequent proof need not exceed a given polynomial function of the endsequent. In order to obtain a complete contraction free fragment of the sequent calculus for **S4**, Mints's translation of modal formulae into modal clauses is used. Mints's notion of modal clause is also used to provide polynomial translations of **S4** and **K4** into **K**, by means of a preliminary (polynomial) rewriting of the input formulae into clausal form.

Keywords: Modal Logics, Automated Deduction, Sequent Calculi, Tableau Calculi.

Résumé. Ce papier est une étude de quelques méthodes de déduction automatique pour des logiques modales propositionnelles. Nous utilisons des systèmes de preuve par tableaux pour montrer que la traduction de la logique modale **S4** en **T** proposée par Fitting peut être construite en temps déterministe polynômial. En plus, nous établissons une borne polynômiale sur la longueur des branches au cours de la recherche de preuves pour les logiques transitives **S4** and **K4**. Ceci permet d'éliminer les tests de périodicité quand on prouve la validité en **S4**; il constitue aussi une sorte d'"élimination des contractions" pour les calculs des séquents modaux, dans le sens où le nombre de contractions nécessaires sur une branche d'une preuve par séquents n'a pas besoin de dépasser une certaine fonction polynômiale du séquent conclusion. Afin d'obtenir un fragment sans contractions et complet du calcul des séquents pour **S4**, nous utilisons la mise en forme clausale modale proposée par Mints. La notion de clause modale de Mints est aussi utilisée pour construire des traductions polynômiales de **S4** et **K4** en **K**, à l'aide d'une réécriture préliminaire des formules sous forme de clauses.

Mots-clé: Logiques Modales, Déduction Automatique, Calculs des Séquents, Calculs par Tableaux.

C.R.: F.2.2, F.4.1, I.2.3

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1 Introduction

Several sound and complete proof-systems for a variety of modal logics have been defined, that lead naturally to the design of modal theorem provers: tableau calculi [6, 7], sequent calculi [6, 11, 12], resolution systems [1, 5, 11].

Among modal logics, we distinguish "transitive logics", where the schema $\Box A \rightarrow \Box\Box A$ holds, and non-transitive ones. In particular, we shall consider **S4** and its non-transitive counterpart, **T**. Although the validity problem has the same structural complexity for the two above mentioned logics (it is PSPACE complete), the design of "relatively feasible" automated proof-procedure is considerably more complex in the case of **S4** versus **T**. Let us consider, for example, modal tableau calculi. For the non transitive logic **T**, at any step in the expansion of a tableau there is only a finite number of choices and the construction of any tableau terminates. In fact, although the number of logical operators may not be reduced by the application of a ν-rule (see Section 1.1 for a formulation of the modal tableau rules), it decreases as soon as a π-rule is applied; if a node contains no π-formulae, and, in any case, before the application of a π-rule, reapplication of ν-rules can be avoided. Therefore, the number of tableaux for a given set of formulae is finite and the length of each branch in one of them is polynomially bounded in the number of logical symbols in the initial set. The case of the transitive logic **S4** is not so simple. In fact, there exist infinite tableaux for a given set of formulae. However, the subformula property enjoyed by tableau calculi ensures that any infinite tableau cycles, so that its depth can be bound by 2^k, where k is the number of subformulae in the initial set. So, in principle, a tableau based decision method can be obtained by inclusion of a periodicity test. A different approach is proposed by Fitting [7], who defines mappings to translate **S4**, **K4** and **T** into the weakest normal modal logic **K**. Such mappings take two arguments: a set Σ of modal formulae and a positive integer n; they give as value a new set $map(\Sigma, n)$ of modal formulae. The mappings enjoy the following property: Σ is contradictory in **S4** (resp. **K4**, **T**) if and only if *there exists* an integer n such that $map(\Sigma, n)$ is contradictory in **K**.

The advantage of this approach is not only the reduction of the satisfiability problem for non-transitive logics to the simpler analogous problem for **K**, but also the possibility of obtaining theorem-provers for the considered logics from a "kernel" theorem-prover for **K**. This holds, of course, not only for tableau based systems, but also for other proof methods, such as, for example, resolution based ones.

However, in [7] no way to compute an adequate value of the second argument of the mappings, namely n, given the first argument Σ, is provided. Actually, the PROLOG theorem provers for the transitive logics given in the appendix of [7] take a positive integer n as additional input. In [4], where a resolution proof procedure for **S4** is defined, it is remarked that $n = 2^k$, where k is the number of subformulae of Σ, will do. This upper bound, as we shall see, follows directly from the subformula property enjoyed by sequent and tableau calculi. To the best of our knowledge, no better way to compute n out of Σ has been pointed out.

In this work we show the existence of a polynomial function f of some measure m of the size of Σ, such that $n = f(m)$. This result does not furnish, directly, a polynomial translation of the logics **S4** and **K4** into **K**: the fact that **K** is not reflexive forces the translation to recursively multiply the modal operators and the formulae in their scope, in such a way that, even if n is a polynomial function of the input set Σ, the translation $map(\Sigma, n)$ cannot be polynomially bounded in the length of Σ.

In order to overcome this fact, we consider the intermediate translation of **S4** into **T**; in this case, the resulting formulae are provably polynomial in the length of the input. As a consequence, the periodicity tests specific to **S4** can be avoided by means of the polynomial translation in **T** of the formula to be proved. Moreover, a byproduct of the technique used to prove our result is the possibility of defining a complete strategy for proof search *in* **S4** such that periodicity tests are replaced by the construction of tableaux whose height is polynomially bounded.

The existence of the above polynomial function f is essentially proved by showing that (i) f depends on the maximum number of applications of the π-rule needed in a branch of a closed **S4** tableau for Σ, and that (ii) a polynomial bound to the number of applications of the π-rule in a branch of a tableau refutation exists. To be fair, it must be said that the key observation needed to prove (ii) is already used in the proof of the computational complexity of the validity problem in **S4** given by Ladner in [10] (see also [8]). However, to the best of our knowledge, this fact seems not to have been exploited in the literature on modal automated deduction. We confess that, facing the problem of defining a natural polynomial translation of **S4** into **T** (the existence of a polynomial reduction being guaranteed by the fact that the satisfiability problem in both systems is known to be PSPACE-complete), we found the solution by proving (ii) and, only in a second time, we checked it to be implicit in Ladner's proof.

A second issue addressed in this work is how to eliminate unnecessary duplications of formulae in **S4** proof-search. In order to investigate this problem, we shift from tableau calculi – that we use to study the translation issues – to sequent calculi. Indeed, unnecessary duplications correspond to unnecessary contractions in sequent proofs.

The work is organized as follows. In Section 1 the basic syntax of the modal language is introduced, as well as the rules of the unsigned tableau systems for **K**, **T**, **K4** and **S4**. In Section 2 we define a polynomial translation of **S4** into **T** and show its soundness and completeness. Then we briefly show how to export the above result to **K4**. In Section 3 some consequences concerning proof search in **S4** systems are considered. In particular, in 3.2, modal sequent calculi are introduced, and it is shown that, as a consequence of the previous results, contractions can be eliminated from **S4** proofs. Section 4 concludes this work.

1 Preliminaries

1.1 Tableau Systems for Modal Logics

The propositional modal language extends the classical language by means of the modal operator $\Box$. We consider classical propositional formulae built by use of the connectives $\neg, \vee, \wedge$. Modal formulae are obtained by addition of the following clause to the usual inductive definition of propositional formulae:

- If A is a formula, then $\Box A$ is a formula.

The modal operator $\Diamond$ is usually a defined symbol: $\Diamond A =_{Def} \neg\Box\neg A$. In the sequel, the expression "boxed formula" will be used to denote a formula of the form $\Box A$, while a "non-boxed formula" is a formula whose main logical operator (if any) is not $\Box$.

The modal propositional system **K** is axiomatically characterizable by addition of the following schema and inference rule to classical propositional axioms and rules:

(K) $\Box(A \rightarrow B) \rightarrow (\Box A \rightarrow \Box B)$

(Nec) $\dfrac{A}{\Box A}$

The systems **T**, **K4** and **S4** are the extensions of **K** obtained by addition, respectively, of the schema (T), the schema (4), and both the schemata (T) and (4) below.

(T) $\Box A \rightarrow A$

(4) $\Box A \rightarrow \Box\Box A$

The above systems can be given sound and complete tableau calculi [6]. We choose to work with unsigned tableaux and take negation over non-atomic formulae as a defined symbol. This preserves compactness in the formulation of the rules, without hindering the reader not accustomed with Fitting's notation. As a further advantage, we need not use the notion of positive and negative occurrences of a subformula.

In our syntax, modal formulae are then built out of literals (atoms and negated atoms) by use of $\wedge$, $\vee$, $\Box$ and $\Diamond$. Negation over non-atomic formulae is defined as usual:

$$\neg(A \wedge B) =_{Def} \neg A \vee \neg B, \qquad \neg(A \vee B) =_{Def} \neg A \wedge \neg B,$$

$$\neg\neg A =_{Def} A, \qquad \neg\Box A =_{Def} \Diamond\neg A, \qquad \neg\Diamond A =_{Def} \Box\neg A$$

The unsigned tableau rules can thus be reformulated as follows. Below, Γ is a set of modal formulae, $\Box\Gamma$ denotes the set $\{\Box G \mid G \in \Gamma\}$, Δ is a set of non-boxed modal formulae, comma is set union. The two classical rules are:

$$(\alpha)\quad \frac{A \wedge B, \Gamma}{A, B, \Gamma} \qquad\qquad (\beta)\quad \frac{A \vee B, \Gamma}{A, \Gamma \qquad B, \Gamma}$$

The tableau systems for the four logics we are considering are obtained by addition of the following rules to the above ones.

K		$(\pi_K)\quad \dfrac{\Diamond A, \Box\Gamma, \Delta}{A, \Gamma}$
K4		$(\pi_4)\quad \dfrac{\Diamond A, \Box\Gamma, \Delta}{A, \Box\Gamma, \Gamma}$
T	$(\nu_T)\quad \dfrac{\Box A, \Gamma}{A, \Box A, \Gamma}$	$(\pi_K)\quad \dfrac{\Diamond A, \Box\Gamma, \Delta}{A, \Gamma}$
S4	$(\nu_T)\quad \dfrac{\Box A, \Gamma}{A, \Box A, \Gamma}$	$(\pi_{S4})\dfrac{\Diamond A, \Box\Gamma, \Delta}{A, \Box\Gamma}$

When tableau rules are compared with sequent calculi rules [6], it is apparent that a contraction is implicit in the rule (ν_T), a sequence of weakenings is implicit in the rules (π_K) and (π_{S4}), and both contractions and weakenings are implicit in (π_4).

This formulation of the calculi enjoys the *subformula property*: if $\mathcal{T}$ is a tableau for the set of modal formulae Σ, then every formula in a node in $\mathcal{T}$ is a subformula of a formula in Σ. As a consequence, every node in a tableau for Σ is a subset of the set of subformulae in Σ; hence, any refutable set Σ of formulae has a closed tableau whose depth does not exceed 2^n, where n is the number of subformulae in Σ.

We shall use the expressions **S**-tableau, **S**-contradictory, etc., where **S** is any of **K**, **K4**, **T** or **S4**, in the obvious sense. In the proofs of our results, we will use the following obvious property:

Remark 1 *Let* **S** *be any of* **K**, **K4**, **T** *or* **S4**. *If a set of modal formulae Σ is* **S***-contradictory, the formula $A \rightarrow B$ is* **S***-valid and Σ' is obtained from Σ by substituting A for some occurrences of B which are not in the scope of a negation, then Σ' is still* **S***-contradictory.*

1.2 Mints Modal Clauses

In the sequel of the paper, we will make use of the notion of *modal clause* introduced by Mints in [11]. Here, we recall the definition of such a notion.

Definition 1 (Mints modal clauses) *A modal literal is a formula having one of the following forms: $p, \neg p, \Box p, \Diamond p$, where p is atomic. A modal clause is a formula having either the form $L_1 \vee \cdots \vee L_n$ or the form $\Box(L_1 \vee \cdots \vee L_n)$, where each L_i is a modal literal. A modal clausal sequent is a sequent whose formulae are modal clauses.*

Theorem 1 (Mints) *Let F be a modal formula. There are a finite set of modal clauses C_F and a new propositional variable p such that F is* **S4***-valid if and only if the formula $A \rightarrow p$ is* **S4***-valid, where A is the conjunction of the elements in C_F.*

The essential idea in Mints' transformation into clausal form is to replace non-atomic subformulae of the original formula F by new propositional variables (each subformula A being associated to a new variable p_A) and build up clauses declaring that each new variable is equivalent to the associated subformula. For instance, if the subformula A has the form $\neg B$, and p_A, p_B are the variables associated respectively to A and B, the clausal form of F contains the following clauses:

$$\Box(\neg p_A \vee \neg p_B), \Box(p_A \vee p_B)$$

An immediate consequence of Theorem 1 is that, given any finite set Γ of modal formulae, there exists a finite set Γ^c of modal clauses such that Γ is **S4**-unsatisfiable if and only if Γ^c is so.

2 Translations

2.1 A Polynomial Translation of S4 into T

In this section we give a reformulation of Fitting's mapping $\mathcal{M}_{S4,T}$ of the transitive logic **S4** into **T**, and show that if Σ is an **S4**-refutable set of formulae, then the set $\mathcal{M}_{S4,T}(\Sigma, n)$ is **T**-refutable, where n is the maximum number of applications of the π_{S4} rule in a closed tableau for Σ. Then a polynomial bound on the number of applications of the π_{S4} rule in a closed tableau for Σ will be established, along the lines of [10], thus leading to a natural (sound and complete) polynomial translation of **S4** into **T**.

In the sequel, the notation $\Box^n$ is used to denote a sequence of n occurrences of the $\Box$ operator, with the convention that $\Box^0 A = \Box^1 A = \Box A$.

Definition 2 *Let F be a modal formula, Σ a set of modal formulae, and n a natural number.*

- *The mapping $\mathcal{M}_{S4,T}(F,n)$ is inductively defined as follows:*
  - $\mathcal{M}_{S4,T}(L,n) = L$ *if* L *is a literal;*
  - $\mathcal{M}_{S4,T}(A \wedge B, n) = \mathcal{M}_{S4,T}(A,n) \wedge \mathcal{M}_{S4,T}(B,n)$*;*
  - $\mathcal{M}_{S4,T}(A \vee B, n) = \mathcal{M}_{S4,T}(A,n) \vee \mathcal{M}_{S4,T}(B,n)$*;*
  - $\mathcal{M}_{S4,T}(\Box A, n) = \Box^n \mathcal{M}_{S4,T}(A,n)$*;*
  - $\mathcal{M}_{S4,T}(\Diamond A, n) = \Diamond \mathcal{M}_{S4,T}(A,n)$*.*
- $\mathcal{M}_{S4,T}(\Sigma,n) = \{\mathcal{M}_{S4,T}(F,n) \mid F \in \Sigma\}$*.*

In the sequel, we shall use the shorthands F^n and Σ^n for $\mathcal{M}_{S4,T}(F,n)$ and $\mathcal{M}_{S4,T}(\Sigma,n)$, respectively.

The following lemmas constitute the essential steps to obtain our result.

Lemma 1 *Let Γ be a set of modal formulae and n a natural number. If there is a closed* **S4***-tableau $\mathcal{T}$ for Γ such that the maximum number of applications of the π_{S4} rule in any branch of $\mathcal{T}$ is less than or equal to n, then there is a closed* **T***-tableau $\mathcal{T}'$ for Γ^n.*

Proof. By induction on $\mathcal{T}$.

1. *(Base)* The set Γ is a closed leaf $\Lambda, p, \neg p$. Trivially, $\Lambda^n, p, \neg p$ is still a closed leaf.
2. *(Inductive step)* We distinguish several cases according to the form of the first inference in $\mathcal{T}$. We show here the treatement of the modal rules and the β-rule. The other classical rules are treated similarly.
   - The first applied rule is π_{S4}. In this case, Γ has the form $\Diamond A, \Box\Lambda, \Delta$, where Δ is a set of non-boxed formulae, and Γ^n is $\Diamond A^n, \Box^n\Lambda^n, \Delta^n$. The **S4**-tableau $\mathcal{T}$ has the form:

     $$(\pi_{S4}) \quad \frac{\Diamond A, \Box\Lambda, \Delta}{\begin{array}{c} A, \Box\Lambda \\ [\mathcal{T}_0] \end{array}}$$

     Since the number of applications of the π_{S4} rule in a branch of $\mathcal{T}_0$ is bounded by $n-1$, by the inductive hypothesis there exists a closed **T**-tableau $\mathcal{T}_0'$ for the set $A^{n-1}, \Box^{n-1}\Lambda^{n-1}$. Since $\Box\Box F \rightarrow \Box F$ is **T**-valid, any set obtained from a refutable set replacing some occurrences of a subformula $\Box F$ (recall that such occurrences are always "positive") with $\Box\Box F$ is still refutable (see the remark 1). Therefore, there is a closed **T**-tableau $\mathcal{T}_0''$ of the set $A^n, \Box^{n-1}\Lambda^n$. The desired tableau is then:

     $$(\pi_K) \quad \frac{\Diamond A^n, \Box^n\Lambda^n, \Delta^n}{\begin{array}{c} A^n, \Box^{n-1}\Lambda^n \\ [\mathcal{T}_0''] \end{array}}$$

- The first applied rule is ν_T, i.e. $\mathcal{T}$ has the form:

$$(\nu_T) \quad \frac{\Box A, \Lambda}{\begin{array}{c} A, \Box A, \Lambda \\ [\mathcal{T}_0] \end{array}}$$

  and $\Gamma^n = \{\Box^n A^n\} \cup \Lambda^n$. By the inductive hypothesis there is a closed **T**-tableau $\mathcal{T}_0'$ of the set $\{A^n, \Box^n A^n\} \cup \Lambda^n$. Hence there is also a closed **T**-tableau $\mathcal{T}_0''$ of the set

$$\{A^n, \Box A^n, \cdots, \Box^{n-1} A^n, \Box^n A^n\} \cup \Lambda^n$$

  So $\mathcal{T}$' is:

$$\begin{array}{c} \dfrac{\Box^n A^n, \Lambda^n}{\Box^{n-1} A^n, \Box^n A^n, \Lambda^n} (\nu_T) \\ \vdots \\ \dfrac{}{A^n, \Box A^n, \cdots, \Box^{n-1} A^n, \Box^n A^n, \Lambda^n} (\nu_T) \\ [\mathcal{T}_0''] \end{array}$$

- The first applied rule is a β rule. The **S4**-tableau $\mathcal{T}$ has then the form:

$$(\beta) \quad \frac{A \vee B, \Lambda}{\begin{array}{cc} A, \Lambda & B, \Lambda \\ [\mathcal{T}_1] & [\mathcal{T}_2] \end{array}}$$

  An application of the β rule to $\Gamma^n = \{A^n \vee B^n\} \cup \Lambda^n$ produces two sets, $\{A^n\} \cup \Lambda^n$ and $\{B^n\} \cup \Lambda^n$. Since in both $\mathcal{T}_1$ and $\mathcal{T}_2$ the number of applications of the π rule is bounded by n, by the inductive hypothesis there exist two closed **T**-tableaux $\mathcal{T}_1'$ and $\mathcal{T}_2'$ for $\{A^n\} \cup \Lambda^n$ and $\{B^n\} \cup \Lambda^n$, respectively. The tableau $\mathcal{T}$' is then:

$$(\beta) \quad \frac{A^n \vee B^n, \Lambda^n}{\begin{array}{cc} A^n, \Lambda^n & B^n, \Lambda^n \\ [\mathcal{T}_1'] & [\mathcal{T}_2'] \end{array}}$$

■

Remark 2 *By analyzing the syntactical operations leading to the construction of the closed* **T***-tableau* $\mathcal{T}'$ *for* Γ^n *from the* **S4***-tableau* $\mathcal{T}$ *for* Γ *in the above lemma, something can be said about the length of* $\mathcal{T}'$*: if* k *is the height of* $\mathcal{T}$ *and* p *is the length of* Γ*, then the height of* $\mathcal{T}'$ *is bounded by* $k + k \times n \times p^2$*, where* n *is as in the statement of the Lemma.*

However, a different and simpler reasoning leads to bound the height of $\mathcal{T}'$ by $(p \times n)^2$. In fact, the length of Γ^n is bounded by $p \times n$ and the height of any **T***-tableau is bounded by the square of the length of its root.*

Note also that, as a consequence of lemma 1, the subformula property and the obvious fact that if Γ^n is **T***-contradictory, then Γ is* **S4***-contradictory, we get what already observed in [4], that is the possibility of taking $n = 2^s$, where s is the number of subformulae of Γ, as the second argument for Fitting's mapping $\mathcal{M}_{S4,T}$.*

The following lemma establishes a linear bound to the number of applications of the π_{S4}-rule *on a given formula* in an **S4**-tableau for Σ; in fact, it can be bounded by the number of boxed subformulae in Σ (plus one).

Lemma 2 *Let Σ be a set of modal formulae that contains p boxed subformulae, and let $\mathcal{T}$ be a closed* **S4***-tableau for Σ. Then there is a closed* **S4***-tableau $\mathcal{T}^*$ for Σ such that the π_{S4}-rule is applied at most $p+1$ times to the same formula in every branch.*

Proof. Let us assume that there is a branch $\mathcal{B}$ in $\mathcal{T}$ containing $p+k$ applications of the π_{S4}-rule to the same formula $\Diamond A$, where $k \geq 2$. We show how to eliminate at least one application of the π_{S4}-rule to $\Diamond A$ from $\mathcal{B}$ – and possibly some sub-branches steaming from $\mathcal{B}$. The tableau $\mathcal{T}^*$ is then obtained by repeated application of such a transformation.

Let $\Gamma = \{B \mid \Box B \text{ a subformula in } \Sigma\}$. Note that:

1. Every application $\mathcal{J}_i$ of the π_{S4}-rule to $\Diamond A$ in $\mathcal{T}$ has the form:

$$\frac{\Diamond A, \Box\Gamma_i, \Delta_i}{A, \Box\Gamma_i}$$

   where $\Gamma_i \subseteq \Gamma$.

2. If $\mathcal{J}_i$ and $\mathcal{J}_j$ are applications of the π_{S4}-rule to $\Diamond A$ in $\mathcal{B}$, and $\mathcal{J}_i$ is above $\mathcal{J}_j$:

$$\begin{array}{c} \vdots \\ (\mathcal{J}_1)\ \dfrac{\neg\Box A_i, \Box\Delta_i, \Lambda_i}{\neg A_i, \Box\Delta_i} \\ \vdots \\ (\mathcal{J}_2)\ \dfrac{\neg\Box A_j, \Box\Delta_j, \Lambda_j}{\neg A_j, \Box\Delta_j} \\ \vdots \end{array}$$

then $\Gamma_i \subseteq \Gamma_j$.

Item (1) follows immediately from the subformula property. Item (2) is proved by showing that for every inference $\dfrac{\Psi}{\Psi_0}$ and $\dfrac{\Psi}{\Psi_0 \quad \Psi_1}$ in $\mathcal{T}$:

$$\{B \mid \Box B \in \Psi\} \subseteq \{B \mid \Box B \in \Psi_i\} \text{ for } i = 0, 1$$

The proof of this fact is an easy induction on $\mathcal{T}$.

Therefore, the branch $\mathcal{B}$ has the form:

$$\begin{array}{cc}
 & \Sigma \\
 & \vdots \\
(\mathcal{J}_1) & \dfrac{\Diamond A, \Box\Gamma_1, \Delta_1}{A, \Box\Gamma_1} \\
 & \vdots \\
(\mathcal{J}_2) & \dfrac{\Diamond A, \Box\Gamma_2, \Delta_2}{A, \Box\Gamma_2} \\
 & \vdots \\
(\mathcal{J}_{p+1}) & \dfrac{\Diamond A, \Box\Gamma_{p+1}, \Delta_{p+1}}{A, \Box\Gamma_{p+1}} \\
 & \vdots \\
(\mathcal{J}_{p+k}) & \dfrac{\Diamond A, \Box\Gamma_{p+k}, \Delta_{p+k}}{A, \Box\Gamma_{p+k}} \\
 & \vdots
\end{array}$$

where $\mathcal{J}_1, \mathcal{J}_2, ..., \mathcal{J}_{p+1}, ..., \mathcal{J}_{p+k}$ are all the (consecutive) applications of the π_{S4}-rule to $\Diamond A$ in $\mathcal{B}$, and $\Gamma_1 \subseteq \Gamma_2 \subseteq ... \subseteq \Gamma_{p+1} \subseteq ... \subseteq \Gamma_{p+k} \subseteq \Gamma$.

Since $|\Gamma| = p$, there are at least two sets Γ_i and Γ_j such that $\Gamma_i = \Gamma_j$. Let us assume in fact that Γ_1 contains m formulae ($m \geq 0$) and that, for all $h = 1, ..., p - m$, Γ_h is different from Γ_{h+1}. Then Γ_{h+1} contains at least one more formula w.r.t. Γ_h, so that Γ_{p-m+1} must be equal to Γ; actually,$\Gamma_h = \Gamma$ for all $h > p - m$.

Let then $\Gamma_i = \Gamma_j$. So the branch has the following form:

$$\Sigma$$

$$\begin{array}{c} \cdot \\ \cdot \\ \cdot \\ (\mathcal{J}_i)\ \dfrac{\Diamond A, \Box\Gamma_i, \Delta_i}{A, \Box\Gamma_i} \\ [\mathcal{T}_0] \\ (\mathcal{J}_j)\ \dfrac{\Diamond A, \Box\Gamma_i, \Delta_j}{A, \Box\Gamma_i} \\ [\mathcal{T}_1] \end{array}$$

It is clear that the part $\mathcal{T}_0$ of the tableau $\mathcal{T}$ can be dispensed with and $\mathcal{T}$ can be transformed into:

$$\begin{array}{c} \Sigma \\ \cdot \\ \cdot \\ \cdot \\ (\mathcal{J}_i)\ \dfrac{\Diamond A, \Box\Gamma_i, \Delta_i}{A, \Box\Gamma_i} \\ [\mathcal{T}_1] \end{array}$$

The tableau that is obtained is closed and the corresponding branch has at least one application of the π_4-rule on $\Diamond A$ less than $\mathcal{T}$. ∎

As a direct consequence of the subformula property and Lemma 2, we have the following

Lemma 3 *Let Σ be a set of modal formulae that contains p boxed subformulae and n subformulae of the form $\Diamond A$. If Σ is* **S4***-contradictory, then there is a closed* **S4***-tableau for Σ where every branch contains at most $n \times (p+1)$ applications of the π_{S4}-rule, hence at most m^2 applications of the π_{S4}-rule, where $m = n + p$.*

The existence of a polynomial translation of **S4** into **T** can be established:

Theorem 2 *Let Σ be a set of modal formulae that contains p boxed subformulae and n subformulae of the form $\Diamond A$. The set Σ is* **S4***-contradictory if and only if $\mathcal{M}_{S4,T}(\Sigma, (n \times (p+1))^2)$ is* **T***-contradictory, hence if and only if $\mathcal{M}_{S4,T}(\Sigma, m^2)$ is* **T***-contradictory, where $m = n + p$*

Proof. The "if" part is trivial: **T** is included in **S4** and $\Box A$ and $\Box\Box A$ are **S4**-equivalent. For the converse, it suffices to combine Lemmas 1 and 3. ∎

As an easy example of application of the above theorem, consider the set $\{\Box p, \Diamond\Diamond\neg p\}$, where p is a propositional letter. This set is typically **S4**-contradictory but **T**-satisfiable. It contains one $\Box$ and 2 $\Diamond$'s, so any set of the form $\{\Box^m p, \Diamond\Diamond\neg p\}$, where $m \geq 9$ is **T**-refutable. Actually, $\{\Box^2 p, \Diamond\Diamond\neg p\}$ is also **T**-refutable, because there is an **S4**-refutation of $\{\Box p, \Diamond\Diamond\neg p\}$ using the π_{S4}-rule no more than twice in every branch (see Lemma 1).

Note that the mapping $\mathcal{M}_{S4,T}(\Sigma, n)$ could be combined with Fitting's mapping from **T** into **K** (see [7]), so as to get a translation of **S4** directly into **K**. However, the length of the final output would not necessarily be polynomially bounded in the length of the input set, because the mapping from **T** into **K** is exponential in the number of modal operators of its input formulae.

2.2 What about K4?

In this section we consider which among the results already established for **S4** still hold and are meaningful for the transitive modal logic **K4**.

The next definition is a reformulation of Fitting's mapping from **K4** into **K** [7].

Definition 3 *Let F be a modal formula, Σ a set of modal formulae, n a natural number.*

- *The mapping $\mathcal{M}_{K4,K}(F, n)$ is defined by induction on F:*
  - $\mathcal{M}_{K4,K}(L, n) = L$, *if L is a literal;*
  - $\mathcal{M}_{K4,K}(A \wedge B, n) = \mathcal{M}_{K4,K}(A, n) \wedge \mathcal{M}_{K4,K}(B, n)$;
  - $\mathcal{M}_{K4,K}(A \vee B, n) = \mathcal{M}_{K4,K}(A, n) \vee \mathcal{M}_{K4,K}(B, n)$;
  - $\mathcal{M}_{K4,K}(\Box A, n) = \Box\mathcal{M}_{K4,K}(A, n) \wedge \cdots \wedge \Box^n\mathcal{M}_{K4,K}(A, n)$;
  - $\mathcal{M}_{K4,K}(\Diamond A, n) = \Diamond\mathcal{M}_{K4,K}(A, n)$
- $\mathcal{M}_{K4,K}(\Sigma, n) = \{\mathcal{M}_{K4,K}(F, n) \mid F \in \Sigma\}$.

In the sequel, we use the notations $A(n)$ and $\Sigma(n)$ as shorthands for, respectively, $\mathcal{M}_{K4,K}(A, n)$ and $\mathcal{M}_{K4,K}(\Sigma, n)$.

The existence of a translation of **K4** into **K** can be proved by use of intermediate results analogous to Lemmas 1, 2 and 3.

Lemma 4 *Let Γ be a set of modal formulae and n a natural number. If there is a closed* **K4***-tableau $\mathcal{T}$ for Γ such that the maximum number of applications of the π_4 rule in any branch of $\mathcal{T}$ is less than or equal to n, then there is a closed* **K***-tableau for $\Gamma(n)$.*

Proof. As in the case of Lemma 1, the proof is an easy induction on $\mathcal{T}$. We show here the sample case of the induction step, where the first rule applied in $\mathcal{T}$ is π_4. Here Γ has the form $\Diamond A, \Box\Lambda, \Delta$, where Δ contains no boxed formula. Say that $\Lambda = \{B_1, \cdots, B_p\}$. Thus, $\mathcal{T}$ has the form

$$\frac{\Diamond A, \Box B_1, \cdots, \Box B_p, \Delta}{A, \Box B_1, \cdots, \Box B_p, B_1, \cdots B_p}$$
$$[\mathcal{T}_0]$$

Applying the map $\mathcal{M}_{K4,K}$ to Γ we get the set

$$\{\Diamond A(n)\} \cup \{\Box B_i(n) \wedge \cdots \wedge \Box^n B_i(n) \ \mid B_i \in \Lambda\} \cup \Delta(n) \quad (1)$$

Since the number of applications of the π_4-rule in a branch of $\mathcal{T}_0$ is bounded by $n-1$, by the inductive hypothesis there is a closed **K**-tableau for

$$\{A(n-1)\} \cup \{\Box B_i(n-1) \wedge \cdots \wedge \Box^{n-1} B_i(n-1) \ \mid B_i \in \Lambda\} \cup \Lambda(n-1) \quad (2)$$

By using Remark 1 and the trivial fact that $(A \wedge B) \rightarrow A$ is a classical tautology, we get the existence of a closed **K**-tableau for the set

$$\{A(n)\} \cup \{\Box B_i(n) \wedge \cdots \wedge \Box^{n-1} B_i(n) \ \mid B_i \in \Lambda\} \cup \Lambda(n) \quad (3)$$

Since a set having the form $\Gamma \cup \{A \wedge B\}$ is refutable if and only if the set $\Gamma \cup \{A, B\}$ is so, this implies the existence of a closed **K**-tableau $\mathcal{T}'_0$ for the set

$$\{A(n)\} \cup \{B_1(n), \cdots, \Box^{n-1} B_1(n)\} \cdots \cup \{B_p(n), \cdots, \Box^{n-1} B_p(n)\} \quad (4)$$

Now, from the set 1, by using first α-rules, then the π_K rule we obtain exactly the set 4. Hence it suffices to continue the tableau by appending $\mathcal{T}'_0$. ∎

Lemma 5 *Let Σ be a set of modal formulae that contains p boxed subformulae, and let $\mathcal{T}$ be a closed* **K4***-tableau for Σ. Then there is a closed* **K4***-tableau T^* for Σ such that the π_4-rule is applied at most $p+1$ times to the same formula in every branch.*

Proof. The proof of Lemma 2 is correct also when the π_{S4}-rule is replaced by the π_4-rule. ∎

The above lemma and the subformula property give the following analogous of Lemma 3:

Lemma 6 *Let Σ be a set of modal formulae that contains p boxed subformulae and n subformulae of the form $\Diamond A$. If Σ is* **K4***-contradictory, then there is a closed* **K4***-tableau for Σ where every branch contains at most $n \times (p+1)$ applications of the π_4-rule, hence at most m^2 applications of the π_{K4}-rule, where $m = n + p$.*

By using the obvious fact that in **K4** a formula $\Box A \wedge \cdots \wedge \Box^n A$ is equivalent to $\Box A$ and Lemmas 4, 6, we can therefore establish the following result:

Theorem 3 *Let Σ be a set of modal formulae that contains p boxed subformulae and n subformulae of the form $\Diamond A$. Let m be $n+p$. The set Σ is* **K4**-*contradictory if and only if $\mathcal{M}_{K4,K}(\Gamma, m^2)$ is* **K**-*contradictory.*

Although Theorem 3 is a result analogous to Theorem 2, the fact that **K** is not reflexive forces the translation to recursively multiply the modal operators and the formulae in their scope, in such a way that, even if the parameter n of the mapping from **K4** to **K** can be bounded by a polynomial function of the length of Σ, the length of $\mathcal{M}_{K4,K}(\Sigma, n)$ cannot be polynomially bounded in the length of Σ.

However, Mints'notion of "modal clause" (see Section 1.2) can be used to provide an alternative translation of **K4** into **K**, polynomial in the size of the input. Note that, given a formula F, the associated set of "Mints clauses" C_F contains a number of logical operators which is a *linear function* of the number of operators in F and it can be constructed in linear time out of F (see [11]). Thus, to obtain a polynomial translation of **K4** into **K**, i suffices to proceed as follows:

1. Polynomially transform the input set of formulae Σ into a set of modal clauses Σ^c;
2. Apply the mapping $\mathcal{M}_{K4,K}$ to Σ^c, with the parameter n equal to the cube of the number of modal operators in Σ^c. The obtained set of formulae Σ' constitutes the translation of Σ into **K**. The length of Σ' is polynomial in the size of Σ, since the application of $\mathcal{M}_{K4,K}$ to a set C of modal *clauses* is polynomial whenever the integer parameter is a polynomial of the size of C.

A similar approach can be used to obtain a polynomial translation of **S4** into **K**.

3 Proof Search in S4

In this section we consider some consequences of the results in Section 2.1 in performing proof search *in* the system **S4**. The reason why an **S4**-tableau can be infinite is that the ν_T rule *systematically* produce a copy of its principal formula $\Box A$. The results established in Section 2.1 allow us to set a polynomial upper bound to the number of times any formula may need to be duplicated. This can be exploited in two respects:

- in the context of automated deduction by means of tableau calculi, a complete strategy for the search of closed **S4**-tableaux can be formulated that does not rely on periodicity tests.

- in the context of sequent calculi, a form of contraction elimination for **S4** is obtainable.

We are now going to clarify these points.

3.1 A Simple Strategy for the Search of Closed S4-Tableaux

As a straightforward consequence of Lemma 3 in automating deduction for **S4** on the basis of tableau calculi, loop-detection can be replaced by the systematic construction of tableaux whose height never exceeds a given polynomial function of the lenght of the imput set of formulae. In fact, tableau construction can be made according to a simple strategy; the following consequence of Lemma 3 is all that is needed.

Lemma 7 *Let Σ be an* **S4***-contradictory set of modal formulae, that contains m modal operators. Then there exists a closed* **S4***-tableau for Σ such that for every branch $\mathcal{B}$,*

1. *If $\mathcal{I}_1$ and $\mathcal{I}_2$ are two applications of the ν_T rule to the same formula in $\mathcal{B}$, then at least one application of the π_{S4} rule is between them.*
2. *$\mathcal{B}$ does not contain more than m^3 applications of the ν_T rule.*

Proof. In order to prove the first item, let $\mathcal{B}$ be a branch in $\mathcal{T}$ containing two application $\mathcal{J}_1$ and $\mathcal{J}_2$ of the ν_T rule on the formula $\Box A$ of this kind:

$$\begin{array}{c}
\Sigma \\
\cdot \\
\cdot \\
\cdot \\
(\mathcal{J}_1)\ \dfrac{\Box A, \Gamma_1}{A, \Box A, \Gamma_1} \\
[\mathcal{T}_0] \\
(\mathcal{J}_2)\ \dfrac{\Box A, \Gamma_2}{A, \Box A, \Gamma_2} \\
[\mathcal{T}_1]
\end{array}$$

where $\mathcal{T}_0$ contains no application of the π_T-rule. We show by induction on $\mathcal{T}_0$ that the inference $\mathcal{J}_2$ can be eliminated.

The base of the induction, where $\mathcal{J}_2$ is immediately below $\mathcal{J}_1$ is obvious. In fact, in that case $\Gamma_1 = \Gamma_2$.

For the inductive step, we consider different cases, according to the form of the inference that is immediately above $\mathcal{J}_2$.

1. $\mathcal{T}_0$ ends with an application of the α-rule, that is immediately followed by $\mathcal{J}_2$, in the form:

$$\cfrac{\cfrac{\Box A, B_0 \wedge B_1, \Gamma_1}{\Box A, B_0, B_1, \Gamma_1}\,(\alpha)}{\Box A, A, B_0, B_1, \Gamma_1}\,\mathcal{J}_2$$

The two displayed inferences can be permuted into:

$$\cfrac{\cfrac{\Box A, B_0 \wedge B_1, \Gamma_1}{\Box A, A, B_0 \wedge B_1, \Gamma_1}\,\mathcal{J}_2}{\Box A, A, B_0, B_1, \Gamma_1}\,(\alpha)$$

The desired result then follows by the inductive hypotehsis.

2. $\mathcal{T}_0$ ends with an application of the β-rule, that is immediately followed by $\mathcal{J}_2$, in either the following form or the symmetric one:

$$\cfrac{\Box A, B_0 \vee B_1, \Gamma_1}{\mathcal{J}_2\ \cfrac{\Box A, B_0, \Gamma_1}{\Box A, A, B_0, \Gamma_1} \qquad \Box A, B_1, \Gamma_1}\,(\beta)$$

The two displayed inferences can be permuted into:

$$\cfrac{\cfrac{\Box A, B_0 \vee B_1, \Gamma_1}{\Box A, A, B_0 \vee B_1, \Gamma_1}\,\mathcal{J}_2}{\Box A, A, B_0, \Gamma_1 \qquad \Box A, A, B_1, \Gamma_1}\,(\beta)$$

Clearly, a closed tableau for $\Box A, A, B_1, \Gamma_1$ can be easily obtained from a closed tableau for $\Box A, B_1, \Gamma_1$. The eliminability of $\mathcal{J}_2$ follows by the inductive hypotehsis.

3. $\mathcal{T}_0$ ends with an application of the ν_T-rule, that is immediately followed by $\mathcal{J}_2$. Without loss of generality, we can suppose that this application of the ν_T-rule is on a formula $\Box B$ where B is different from A, since, as already observed, consecutive applications of the ν_T-rule on the same formula collapse. Hence $\mathcal{T}_0$ ends with:

$$\cfrac{\cfrac{\Box A, \Box B, \Gamma_1}{\Box A, \Box B, B, \Gamma_1}\,(\nu_T)}{\Box A, A, \Box B, B, \Gamma_1}\,\mathcal{J}_2$$

The two displayed inferences can be permuted into:

$$\cfrac{\cfrac{\Box A, \Box B, \Gamma_1}{\Box A, A, \Box B \Gamma_1}\,\mathcal{J}_2}{\Box A, A, \Box B, B, \Gamma_1}\,(\nu)$$

the desired result then follows by the inductive hypotehsis.

Since in $\mathcal{T}_0$ there are no applications of the π_{S4}-rule, the above cases are exhaustive.

To prove the second item, let p be the number of boxed subformulae of Σ and let n be the number of subformulae of the form $\Diamond A$ (hence $m = n + p$). First we observe that if $\mathcal{B}$ contains x applications of the π_{S4}-rule, then, by the first item, $\mathcal{B}$ does not contain more than $x + 1$ applications of the ν_T rule on the same formula. So, $\mathcal{B}$ does not contain more than $p \times (x + 1)$ applications of the ν_T rule. From Lemma 3, x can be bounded by $n \times (p + 1)$, hence the lemma follows. ∎

As a consequence of the above result, in the search of a closed **S4**-tableau for a set of formulae, the branches can be limited polynomially in the length of the input. That is, we have the following result.

Theorem 4 *If Σ is an* **S4***-contradictory set of formulae that has k subformulae, excluding literals (i.e. atoms and negations of atoms), then there exists a closed tableau for Σ where every branch has no more than $(k+1)^3$ inferences.*

Proof. Let n be the number of subformulae of the form $\Diamond A$ in Σ and p the number of boxed subformulae. Lemma 7 shows that if Σ is refutable then there exists a closed tableau $\mathcal{T}$ for Σ built according to the following strategy: first apply as far as possible the classical rules and the ν_T rule once for each formula; this will be called the first block in the branch. Since the application of a classical rule to a formula "consumes" it, the number of possible inferences in such a block is not greater than $k - n$. Then a formula is chosen for an application of the π-rule. After that, classical and ν rules are applied again, no more than once for each formula. In this second block (π-rule followed by the others) there are no more than $k - n + 1$ inferences. The construction leading to the second block is to be repeated at most $n \times (p+1)$ times, according to Lemma 3. Therefore each branch in $\mathcal{T}$ has no more than $k - n + ((k - n + 1) \times n \times (p+1))$ inferences, thus no more than $(k+1)^3$. ∎

A similar reasoning can be applied to **K4**, obviously skipping any reference to ν-rules.

The above proof gives a simple strategy for the search of closed **S4**-tableaux. In [3] several strategies for **S4** proof search are proposed. All of them include some form of periodicity test, which, the author says, remains one of the main factors for the cost of the search. The advantage of the simple strategy that we outline is that it avoids such tests; of course, it is no more than a "kernel", further refinable strategy.

The possibility of avoiding periodicity tests is also a consequence of a result in [9], where a polynomial bound on the length of the branches of proofs in a special sequent calculus for **S4** and a sort of contraction elimination result are established. Such a calculus is specifically devised to deal with sequents having a "normal form" (Mints' clausal form). Theorem 4 shows that a polynomial bound on the height of **S4** proofs for tableaux systems can be established without any need to use normal forms (next subsection applies the same result to sequent systems). Indeed, the approach used in [9] is a *tour de force* through several equivalent formulations of the initial calculus; moreover, a new modal operator is introduced, whose semantics is quite unclear. In contrast, the proof of Theorem 4 is quite simple.

3.2 Contraction Elimination in S4 Sequent Proofs

Our concern here is how to eliminate unnecessary duplications of formulae in **S4** proof search. One of the specific features of sequent calculi is the capability of handling duplications via specific rules, namely contraction rules. Hence, unnecessary duplications correspond to unnecessary applications of contraction rules. For this reason, sequent calculi seem a very appropriate framework to study transformations of proofs into duplication-free proofs. Therefore, we use here a sequent calculus for **S4**.

However, in order to keep close to the "tableau style" used so far and to allow an immediate use of previously established results in the current context, we have chosen a "left handed" formulation of the sequent calculus for **S4**. In this formulation, only left rules are necessary; sequents are expressions of the form $\Gamma \vdash$, where Γ is a multiset of formulae and axioms have the form $p, \neg p \vdash$, where p is an atomic formula. As before, negation over non-atomic formulae is a defined connective.

Below, Γ denotes a multiset of modal formulae and comma is multiset union. $\Box\Gamma$ denotes the multiset of boxed formulae $\{\Box G \mid G \in \Gamma\}$. The logical rules of the sequent calculus for propositional **S4** are:

$$(\wedge\vdash)\frac{\Gamma, A, B \vdash}{\Gamma, A \wedge B \vdash} \qquad\qquad (\vee\vdash)\frac{\Gamma, A \vdash \qquad \Gamma, B \vdash}{\Gamma, A \vee B \vdash}$$

$$(\Box\vdash)\frac{\Gamma, A \vdash}{\Gamma, \Box A \vdash} \qquad\qquad (\Diamond\vdash)\frac{\Box\Gamma, A \vdash}{\Box\Gamma, \Diamond A \vdash}$$

Moreover, there are the two structural rules, *contraction*: $\dfrac{A, A, \Gamma \vdash}{A, \Gamma \vdash}$, and *weakening*: $\dfrac{\Gamma \vdash}{A, \Gamma \vdash}$, and the *cut* rule. The calculus enjoys cut-elimination and throughout the whole section we will consider only cut-free sequent proofs. Obviously such proofs enjoy the subformula property.

It is well known that modal proofs in a "left-handed" formulation of sequent calculus can be mapped to proofs in the more traditional "two-sides" sequent calculus and vice-versa, in analogy with the correspondence between unsigned and signed tableaux. The mapping preserves the number of applications of structural rules.

The strong similarity between tableau refutations and "left-handed" cut-free sequent proofs is apparent. Actually an equivalent formulation of the sequent calculus is possible, where sets of formulae are considered instead of multisets and structural rules are suitably embedded into the logical rules. In such a formulation, a sequent proof is essentially a tableau proof written upside down.

More precisely, the similarity between tableaux and sequent proofs consist in the possibility of getting a sequent proof in correspondence with every closed tableau, essentially inverting the tree, preserving the application of logical rules, possibly adding applications of structural rules and replacing some formulae by a multiplicity of its occurrences. The correspondence being quite straightforward, we just state here the properties that are relevant in this context. Given any closed tableau $\mathcal{T}$ for a set of formulae Γ, there exists a "left-handed" sequent proof $\mathcal{P}$ of $\Gamma \vdash$ such that:

- the maximum number of applications of $(\Box \vdash)$ in a branch of $\mathcal{P}$ is equal to the maximum number of applications of the $\nu_{\mathcal{T}}$-rule in a branch of $\mathcal{T}$;
- an application of the contraction rule systematically follows any application of the $(\Box \vdash)$ rule;
- all the contractions in $\mathcal{P}$ are of this form.

In fact, every application of the $\nu_{\mathcal{T}}$ rule in $\mathcal{T}$, in the form:

$$\frac{\Box A, \Gamma}{A, \Box A, \Gamma}$$

corresponds to the following segment of $\mathcal{P}$:

$$\cfrac{\cfrac{\Gamma, \Box A, A \vdash}{\Gamma, \Box A, \Box A \vdash}\,(\Box \vdash)}{\Gamma, \Box A \vdash}\,(\mathit{contraction})$$

The sequent proofs that enjoy the above properties will be called *canonical*. It is clear that any provable sequent has a canonical proof.

Lemma 7 already provides some sort of "contraction elimination result", in the following sense: the number of contractions necessary in a branch of a canonical **S4** proof need not exceed m^3, where m is the number of modal operators in the endsequent.

However, a natural question arises : can one limit proof-search to the *contraction-free* segment of **S4** sequent calculus, without loosing the expressivity of the whole calculus ?

We answer to this question positively. However, to prove our claim, we need to introduce some additional technical notions.

First, we define a "Fitting style" mapping $\mathcal{M}$ which takes as arguments a multiset of modal formulae and a positive integer n and gives a new multiset of formulae over the same language as a result:

Definition 4 *Let F be a modal formula, Σ a multiset of modal formulae, n a positive integer.*

- *The mapping $\mathcal{M}(F, n)$ is defined by induction on F:*
  - $\mathcal{M}(L, n) = L$ *if L is a literal;*
  - $\mathcal{M}(A \wedge B, n) = \mathcal{M}(A, n) \wedge \mathcal{M}(B, n)$*;*
  - $\mathcal{M}(A \vee B, n) = \mathcal{M}(A, n) \vee \mathcal{M}(B, n)$*;*
  - $\mathcal{M}(\Box A, n) = \underbrace{\Box\mathcal{M}(A, n) \wedge \cdots \wedge \Box\mathcal{M}(A, n)}_{n}$*;*
  - $\mathcal{M}(\Diamond A, n) = \Diamond\mathcal{M}(A, n)$.
- $\mathcal{M}(\Sigma, n) = \{\mathcal{M}(F, n) \mid F \in \Sigma\}$.

In the sequel, we use the notation Σ_n as a shorthand for $\mathcal{M}(\Sigma, n)$.

By exploiting the structure of canonical proofs and Lemma 7, the following result can be proved.

Theorem 5 *Let Σ be a multiset of modal formulae whose length is m. The sequent $\Sigma \vdash$ has an* **S4***-proof if and only if the sequent $\mathcal{M}(\Sigma, m^3) \vdash$ has a contraction-free* **S4***-proof.*

Proof. The "if" direction is immediate.

For the converse implication, let us suppose that $\Sigma \vdash$ is **S4** provable and let $\mathcal{P}$ be a canonical proof for Σ. We make an induction on $\mathcal{P}$ to show that if p is an upper bound on the number of applications of the contraction rule in $\mathcal{P}$, then Σ_p has a contraction-free proof $\mathcal{P}'$. Then the result follows by what already observed: as a consequence of Lemma 7, the number of contractions in a branch of $\mathcal{P}$ need not exceed m^3.

The base is trivial. For the induction step, observe that the last rule in $\mathcal{P}$ cannot be $\Box\vdash$, since $\mathcal{P}$ is canonical, so the cases to consider are the following ones:

- The last rule in $\mathcal{P}$ is a contraction. Then $\mathcal{P}$ has the form:

$$\dfrac{\dfrac{\begin{array}{c}[\mathcal{P}_0]\\ \Gamma, A, \Box A \vdash\end{array}}{\Gamma, \Box A, \Box A \vdash}\ (\Box \vdash)}{\Gamma, \Box A \vdash}\ (contr)$$

The mapping of the endsequent has the form $\Gamma_p, \underbrace{\Box A_p \wedge \cdots \wedge \Box A_p}_{p}$. By the inductive hypothesis, there is a contraction-free proof $\mathcal{P}'_0$ of the mapping of the premise of $\Box \vdash$, i.e. $\Gamma_{p-1}, A_{p-1}, \underbrace{\Box A_{p-1} \wedge \cdots \wedge \Box A_{p-1}}_{p-1} \vdash$. It is easy to show that this implies the existence of a contraction-free proof $\mathcal{P}"_0$ of $\Gamma_p, A_p, \underbrace{\Box A_p \wedge \cdots \wedge \Box A_p}_{p-1}$. An application of $\Box \vdash$ followed by an application of $\wedge \vdash$ gives the desired proof of $\Gamma_p, \underbrace{\Box A_p \wedge \cdots \wedge \Box A_p}_{p}$.

- The last rule in $\mathcal{P}$ is a weakening. A straightforward application of the inductive hypothesis, followed by a weakening, suffices.

- The last rule in $\mathcal{P}$ is $\Diamond \vdash$. In this case, $\mathcal{P}$ has the form:

$$\dfrac{\begin{array}{c}[\mathcal{P}_0]\\ \Box\Gamma, A \vdash\end{array}}{\Box\Gamma, \Diamond A \vdash}\ (\Diamond \vdash)$$

Say that Γ is the multiset $\{G_1, \cdots, G_n\}$. Thus the multi-set $(\Box\Gamma, \Diamond A)_p$ has the form $\{\underbrace{\Box G_{ip} \wedge \cdots \wedge \Box G_{ip}}_{p} \mid 1 \leq i \leq n\} \cup \{\Diamond A_p\}$, where $\cup$ is multiset union. By the inductive hypothesis, there is a contraction-free proof $\mathcal{P}'_0$ of $\{\underbrace{\Box G_{ip} \wedge \cdots \wedge \Box G_{ip}}_{p} \mid 1 \leq i \leq n\} \cup \{A_p\}$. It is easy to show that this implies the existence of a contraction-free proof $\mathcal{P}"_0$ of $\{\underbrace{\Box G_{ip}, \cdots, \Box G_{ip}}_{p} \mid 1 \leq i \leq n\} \cup \{A_p\}$. Then it suffices to apply first $\Diamond \vdash$, then $\wedge \vdash$ several times, to get $(\Box\Gamma, \Diamond A)_p$.

- The last rule in $\mathcal{P}'$ is a classical rule. Here, we consider the sample case of $\vee \vdash$. In this case, $\mathcal{P}$ has the form:

$$\dfrac{\begin{array}{c}[\mathcal{P}_1'] \\ \Gamma, A \vdash\end{array} \qquad\qquad \begin{array}{c}[\mathcal{P}_2'] \\ \Gamma, B \vdash\end{array}}{\Gamma, A \vee B \vdash} \ (\vee \vdash)$$

By the inductive hypothesis, there are contraction-free proofs $\mathcal{P}_1$ and $\mathcal{P}_2$ for, respectively, $\Gamma_p, A_p \vdash$ and Γ_p, B_p. Then it suffices to apply $\vee \vdash$.

∎

The inconvenient with this contraction elimination method is that the length of $\mathcal{M}(\Sigma, m^3)$ cannot be bounded polynomially in the length of Σ. Note that the essential reason of such an exponential complexity is the fact that the depth of the nesting of the $\Box$ operators in a formula containing m occurrences of $\Box$ can be m itself.

For this reason, we propose a modified contraction elimination method, that, again, exploits the notion of Mints modal clause (see Section 1.2). We use the fact that, given any finite set Γ of modal formulae, there exists a finite set Γ^c of modal clauses such that Γ is **S4**-unsatisfiable if and only if Γ^c is so.

Remark 3 *Note that, differently from the general case, if Σ is a set of modal* clauses *(as defined above) and $\mathcal{M}$ is the mapping given in definition 4, then the length of the set of formulae $\mathcal{M}(\Sigma, n)$ is bounded by a* polynomial *of the length of Σ whenever n is also polynomial in the length of Σ. In fact, it is easy to show that, for any n and m greater than 0:*

1. *If c is a modal clause containing m occurrences of the $\Box$ modality, then the number of $\Box$'s in the formula $\mathcal{M}(c, n)$ is bounded by $(n^2 \times (m-1)) + n$.*

2. *For any modal clause c, if n is a polynomial in the length of c, then the length of $\mathcal{M}(c, n)$ is polynomially bounded in c, too.*

The intuitive reason for this is that the maximal depth of the nesting of modal operators in a modal clause c is 2, independently of the number of modalities in c.

This suggests that a finer way to eliminate contractions in **S4** could be to combine theorem 5 with Mints clausal form, namely:

1. Transform the input set of formulae Σ into a corresponding set Σ^c of modal clauses;

2. Translate Σ^c into $\mathcal{M}(\Sigma^c, n)$, for a suitable n polynomially computable out of Σ^c, so as to reduce the problem of the existence of a **S4** sequent proof for Σ to the problem of the existence of a contraction-free **S4** proof of $\mathcal{M}(\Sigma^c, n)$.

Of course, the above method will be feasible only if the "normalization into clausal form" of Σ is itself polynomial. As already observed, this is actually the case. Therefore, the following result is obtained.

Theorem 6 *Let Σ be a multiset of modal formulae containing m modal operators. There exists a modal clausal sequent $\Sigma^c \vdash$ and an integer k which is a polynomial function of the length of Σ such that*

- *The sequent $\Sigma \vdash$ has an* **S4***-proof if and only if the sequent $\mathcal{M}(\Sigma^c, k) \vdash$ has a contraction-free* **S4***-proof.*
- *The length of $\mathcal{M}(\Sigma^c, k)$ is polynomially bounded by the length of Σ and $\mathcal{M}(\Sigma^c, k)$ can be constructed out of Σ in polynomial time.*

4 Concluding Remarks

The contributions of this work to the study of modal automated deduction are the following:

1. It is shown that the integer parameter of Fitting's translations concerning the modal logics **S4**, **K4**, **T**, **K** can be computed polynomially in the size of the input formula.
2. Since the translation of **S4** into **T** is polynomial, an automated proof procedure for **S4**, based either on tableau or sequent calculi, as well as resolution style ones, can be defined by reduction to the simpler case of **T**. This result is of interest also in connection with the automation of different forms of reasoning in modal logic, such as e.g. abduction (see [2]).
3. We show also that in automating deduction *in* **S4** on the basis of tableau calculi, loop-detection can be replaced by the construction of tableaux whose height is bounded polynomially in the size of the input set.
4. As far as proof search in sequent calculi is concerned, the number of contractions to be applied to boxed formulae can be bounded polynomially in the length of the endsequent; alternatively, contractions can be avoided at all, modulo a suitable rewriting of the sequent to be proved.

Determining the existence of a linear function f, allowing us to compute the second argument of Fitting's mappings, is still an open issue. One may be tempted, for example, to guess that in Fitting's translation of **S4** into **T**, the number of times that boxes have to be duplicated in a set Σ is equal to the maximum modal depth of formulae in Σ. But this is false. In fact, consider for example $\Sigma = \{\Box p, \Box q, \Box(\Diamond\neg p \vee r), \Diamond(\Diamond\neg r \vee \neg q)\}$. Σ is **S4**-contradictory and its modal degree is 2. However, $\Sigma^2 = \{\Box\Box p, \Box\Box q, \Box\Box(\Diamond\neg p \vee r), \Diamond(\Diamond\neg r \vee \neg q)\}$ is **T**-satisfiable.

Even if such a linear function f cannot be determined in general, it may be the case that it exists for interesting fragments of **S4**.

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References

[1] M. Abadi and Z. Manna. Modal theorem proving. In *Proc. of the 8th Int. Conf. on Automated Deduction*, pages 172–189, Berlin, 1986. Springer Verlag, LNCS 230.

[2] M. Cialdea Mayer and F. Pirri. Propositional abduction in modal logic. *Journal of the IGPL*, 3:907–919, 1995.

[3] S. Demri. Uniform and non uniform strategies for tableaux calculi for modal logics. *Journal of Applied Non-Classical Logics*, 5:77–96, 1995.

[4] P. Enjalbert and L. Fariñas del Cerro. Modal resolution in clausal form. *Theoretical Computer Science*, 65:1–34, 1989.

[5] L. Fariñas del Cerro. Resolution modal logic. *Logique et Analyse*, 28:153–172, 1985.

[6] M. Fitting. *Proof Methods for Modal and Intuitionistic Logics.* Reidel Publishing Company, 1983.

[7] M. Fitting. First-order modal tableaux. *Journal of Automated Reasoning*, 4:191–213, 1988.

[8] J. Y. Halpern and Y. Moses. A guide to the completeness and complexity for modal logics of knowledge and belief. *Artificial Intelligence*, 54:319–379, 1992.

[9] J. Hudelmaier. On a contraction-free sequent calculus for the modal logic S4. In *Proc. of the 3rd Workshop on Theorem Proving with Analytic Tableaux and Related Methods*, pages 105–112, 1994.

[10] R. E. Ladner. The computational complexity of provability in systems of modal propositional logic. *SIAM Journal of Computation*, 6:467–480, 1977.

[11] G. Mints. Gentzen-type systems and resolution rules. In *International Conference on Computer Logic (COLOG)*, pages 198–231. Springer Verlag, LNCS 417, 1988.

[12] M. Ohnishi and K. Matsumoto. Gentzen method in modal calculi. *Osaka Mathematical Journal*, 9:113–130, 1957.