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Ground and Free-Variable Tableaux for Variants of Quantified Modal Logics*

Abstract. In this paper we study proof procedures for some variants of first-order modal logics, where domains may be either cumulative or freely varying and terms may be either rigid or non-rigid, local or non-local. We define both *ground* and *free variable* tableau methods, parametric with respect to the variants of the considered logics. The treatment of each variant is equally simple and is based on the annotation of functional symbols by natural numbers, conveying some semantical information on the worlds where they are meant to be interpreted.

This paper is an extended version of a previous work where full proofs were not included. Proofs are in some points rather tricky and may help in understanding the reasons for some details in basic definitions.

Keywords: Quantified Modal Logics, Tableaux.

1. Introduction

First-order modal logics constitute a delicate and difficult subject (see for example [14] for an overview). In principle, their semantics could simply be obtained by extension of the modal propositional semantics: a first-order modal structure is built on the base of a set of first-order classical interpretations (the "possible worlds"), connected by a binary relation (the accessibility relation). However, there are some options to be considered, concerning for example:

The object domains: is there any relation between the universes of the different worlds? They can be required to be the same for all worlds (*constant domains*), they can bear no relation one to the other (*varying domains*), or the object domains can vary, but monotonically, i.e. if w' is accessible from w, then the object domain of w is included in the domain of w' (*cumulative domains*).

The designation of terms: is the denotation of ground terms the same in all worlds or can it vary? When the answer is positive, then designation is taken to be *rigid*, otherwise it is *non-rigid* (often, mixed approaches are of interest, too, where some symbols are given a rigid interpretation and others are not).

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The existence of objects: does the extension of any ground term belong to every object domain or not? If the answer is positive, then we assume terms to be *local*, otherwise terms are *non-local*.

So, several variants of quantified modal logics (QMLs) are possible, just by choosing different combinations of the cases considered above. We call them DDE variants (Domains/Designation/Existence variants) of QML. Obviously, the logic also depends on its propositional kernel, so that we have a four-dimensional space of possible QMLs. As often happens with modal logics, different choices are appropriate for different applications. A discussion can be found in [14] as well as [13].

Some DDE variants can easily be given a proof theoretic characterization, some are harder to treat. The easiest approach is obviously obtained just by adding the principles of classical logic to a propositional modal system. What we get is a logic with cumulative domains, rigid designation and local terms. In fact, such a logic validates the converse of Barcan Formula (**CBF**), that characterizes cumulative domains: $\Box\forall x A \rightarrow \forall x \Box A$. Rigid designation and locality of terms are consequences of the instantiation rule of classical logic $\forall x A \rightarrow A[t/x]$, where t is any term. For example, $\Box p(t)$ follows from the instantiation rule and $p(t) \wedge \forall x(p(x) \rightarrow \Box p(x))$, although it is not a logical consequence of the latter formula alone, if the denotation of t can vary from world to world. In fact, in that case, even if $p(t)$ is true at a given world w, i.e. p is true of the object d denoted by t in w, the term t can denote a different object d' in some world w' accessible from w and p may be false for d' at both w and w' (the example is from [18], where $\Box$ is the epistemic "Belief" operator).

Besides the axiomatic characterization, some first-order modal logics with cumulative domains, rigid designation and local terms have been given sequent and tableau calculi [11, 12, 17], natural deduction proof systems [5], matrix proof procedures [21], resolution style calculi [1, 8], and have been treated by means of translation methods [3, 20].

Now, while translation methods are general enough to treat also other DDE variants of QML (for instance, [3] deals with constant domains logics, where designation may be rigid as well as non rigid, and [20] handles all the DDE variants of QML), the direct methods are less flexible. For instance, it can be proved that constant domain logics cannot be treated by standard cut-free sequent calculi ([11], p.383). Note, however that the addition of the classical sequent rules for quantifiers to modal display calculi captures constant domain logics [7], and constant, increasing and decreasing domain modal logics can all be presented as cut-free display sequent calculi, by use of structure-dependent rules for quantifiers [22]. Constant domain logics are also formalizable in the tableau style, but with the addition of prefixes labelling tableau nodes [11], as well as by means of matrix proof methods [21]: in both kinds of calculi in fact it is possible to analyse more than one possible world at a time, and this allows the proof to "go back and forth" (the same mechanism solves the problem of symmetric logics). In [11] all the variants of QML concerning the object domains of possible worlds are treated by prefixed tableau methods and some suggestions on how to treat the case of varying domains by means of calculi without prefixes can be found. A direct approach dealing with both varying domains and non-rigid symbols has a representative in [18], which defines a resolution method for epistemic logics, where terms can be annotated by a "bullet" constructor distinguishing rigid terms from non-rigid ones. A different direction is followed in [13], where the language of modal logic is enriched by means of a predicate abstraction operator, in order to capture differences on the denotation of terms, and a tableau proof procedure is presented for such a logic, with no restriction on the domains of possible worlds.

In general, direct proof methods can treat the less easy variants of QML either by modifications of classical and/or modal rules in an often cumbersome manner (if the axiomatic approach is taken), or else by attaching some kind of semantical information to the proof structures.

The aim of this work is to provide a treatment of some variants of QML, where the explicit intrusion of semantics in the proof structures is kept as simple and minimal as possible. We propose calculi in the tableau style, that allow us to treat all the DDE variants of QML, with the exclusion of constant domain logics. Namely, we deal with the combinations cumulative/varying domains, rigid/non-rigid designation, local/non-local terms, for the analytic logics **K**, **D**, **T**, **K4**, **S4**. The treatment of each DDE variant is equally simple and is based on the annotation of functional symbols by natural numbers, resulting in a minimal departure from standard tableau methods (cut-free sequent systems). A similar mechanism is used in [8, 9], to obtain Skolem-like normal forms and define modal resolution proof procedures. Functional symbols can be annotated by integers, conveying some semantical information on the worlds where they are interpreted. Since in our calculi the annotation is handled dynamically, each node in a tableau, containing the set of active formulae, is also annotated by an integer.

We define both *ground* tableau calculi — called Tab_G (Section 2) — and *free variable* tableau calculi — called Tab_{FV} (Section 3). In both cases, the calculi are parametric with respect to the DDE variants of the considered logic. The treatments differ only in the annotation mechanism and a suitable classification of terms according to the annotation of their symbols, that determines, in Tab_G, which applications of the γ-rule are correct and, in Tab_{FV}, which substitutions are allowed. The calculi in both classes are sound and complete.

This paper is a revised and extended version, including full proofs, of [10]. Although the main features of the calculi Tab_G and Tab_{FV} can look rather intuitive and semantically straightforward, once they are settled and explained, proofs are in some points rather tricky and may help in understanding the reasons of the differences in the basic notions characterizing each of the considered DDE variant of QML.

In the rest of this section, we briefly recall the syntax and semantics of QML. A first-order modal language L is constituted by logical symbols (propositional connectives, quantifiers, modal operators and a countable set X of individual variables), a non-empty set L_P of predicate symbols and a set L_F of functional symbols with an associated arity. Constants are considered as functional symbols with null arity. The inductive definition of terms and formulae is as usual.

A first-order modal interpretation $\mathcal{M}$ of a language L is a tuple $\langle W, w_0, R, D, \delta, \phi, \pi\rangle$ such that:

- W is a non-empty set (the set of "possible worlds");
- w_0 is a distinguished element of W;
- R is a binary relation on W (the *accessibility relation*); wRw' abbreviates $\langle w, w'\rangle \in R$;
- D is a non empty set (the object domain);
- δ is a function assigning to each $w \in W$ a non empty subset of D, the domain of w: $\delta(w) \subseteq D$;
- ϕ represents the interpretation of constants and functional symbols in the language: for every world $w \in W$ and n-ary functional symbol in the language (with $n \geq 0$),
$$\phi(w, f) \in D^n \to D$$
- π is the interpretation of predicate symbols: if p is a n-ary predicate symbol and $w \in W$, then $\pi(w, p) \subseteq D^n$ is a set of n-tuples of elements in D.

The interpretation function ϕ is extended to terms in the usual way, and, by an abuse of notation, $\phi(w, t)$ denotes the interpretation of t in w.

If $\mathcal{M} = \langle W, w_0, R, D, \delta, \phi, \pi\rangle$ is an interpretation of the language L, the **language of the model M**, L_D, is obtained from L by addition of a "name" for each $d \in D$, i.e. a new constant $\overline{d}$. It is assumed that for every $d \in D$ and $w \in W$, $\phi(w, \overline{d}) = d$. Note that the interpretation of names $\overline{d}$ is always rigid.

Below we take $\neg$, $\wedge$ and $\vee$ as the only primitive propositional connectives. The relation $\models$ between an interpretation $\mathcal{M} = \langle W, w_0, R, D, \delta, \phi, \pi \rangle$, a world $w \in W$ and a closed formula in L_D is defined inductively as follows:

1. $\mathcal{M}, w \models p(t_1, ..., t_n)$ iff $\langle \phi(w, t_1), ..., \phi(w, t_n) \rangle \in \pi(w, p)$.
2. $\mathcal{M}, w \models \neg A$ iff $M, w \not\models A$.
3. $\mathcal{M}, w \models A \wedge B$ iff $\mathcal{M}, w \models A$ and $\mathcal{M}, w \models B$.
4. $\mathcal{M}, w \models A \vee B$ iff $\mathcal{M}, w \models A$ or $\mathcal{M}, w \models B$.
5. $\mathcal{M}, w \models \forall x A$ iff for all $d \in \delta(w)$, $\mathcal{M}, w \models A[\overline{d}/x]$
6. $\mathcal{M}, w \models \exists x A$ iff there exists $d \in \delta(w)$ such that $\mathcal{M}, w \models A[\overline{d}/x]$
7. $\mathcal{M}, w \models \Box A$ iff for all $w' \in W$ such that wRw', $\mathcal{M}, w' \models A$
8. $\mathcal{M}, w \models \Diamond A$ iff there is a $w' \in W$ such that wRw' and $\mathcal{M}, w' \models A$

A closed formula A is true in $\mathcal{M}$ iff $\mathcal{M}, w_0 \models A$, and it is valid iff it is true in all interpretations.

The accessibility relation R of a modal structure can be required to satisfy additional properties, characterizing different logics: we consider seriality (**D**), reflexivity (**T**), transitivity (**K4**), both reflexivity and transitivity (**S4**). When no additional assumption on R is made, the logic is called **K**.

On the first-order side, we shall deal with the following DDE variants of QML:

Cumulative domains: for all $w, w' \in W$, if wRw' then $\delta(w) \subseteq \delta(w')$.

Varying domains: no restrictions on $\delta(w)$.

Rigid designation: for all $w, w' \in W$ and $f \in L_F$, $\phi(w, f) = \phi(w', f)$.

Non-rigid designation: no restrictions on the relation between $\phi(w, f)$ and $\phi(w', f)$, for any $w \neq w'$ and $f \in L_F$.

Local terms: for all $w \in W$ and $f \in L_F$, if $d_1, ..., d_n \in \delta(w)$, then $\phi(w, f)(d_1, ..., d_n) \in \delta(w)$.

Non-local terms: no restriction on the closure of domains $\delta(w)$ with respect to the application of the designation of functional symbols.

For the sake of simplicity, we treat each variant as a different logic. However, mixed approaches are also possible. For instance, we can establish that constant symbols denote rigidly and functional symbols (with positive arity) do not. In the calculi described in the rest of this work, each symbol would then be treated according to the rules of the corresponding logic.

Note that, if terms are local, the denotation of every ground term t is in every domain $\delta(w)$. However, if domains are allowed to vary without restrictions, if $\mathcal{M}, w \models \exists x \Diamond p(f(x))$, and d is the element of $\delta(w)$ such that $\mathcal{M}, w \models \Diamond p(f(\overline{d}))$, then the denotation of $f(\overline{d})$ is not necessarily in the domain of the world w' accessible from w and such that $\mathcal{M}, w' \models p(f(\overline{d}))$, unless $d \in \delta(w')$.

As a final remark, we note that if $\phi(w, f)$ is required to be a function from $\delta(w)^n$ to $\delta(w)$, the interpretation of constants is forced to be local. Moreover, the interpretation of a term containing "names" can be undefined in a world, and this fact must be taken into account in the definition of $\models$. Such a definition becomes more complicated also if $\pi(w, p)$ is restricted to the actual domain of the world w, i.e. it is set equal to a subset of $\delta(w)^n$ (see [11], p. 340 ff.).

2. Ground tableaux

In this section we illustrate the mechanism that allows us to deal with the considered DDE variants of QML and present the ground calculi Tab_G. The tableau systems work with formulae in negation normal form: negation over non-atomic formulae is considered as a defined symbol. In other words, modal formulae are built out of literals (atoms and negated atoms) by use of $\wedge$, $\vee$, $\Box$, $\Diamond$ and the quantifiers $\forall$ and $\exists$. Negation over non-atomic formulae is defined as usual.

Formulae occurring in a ground tableau for a set S of modal formulae are built over an extension L^+ of the language L of S, where constants and functional symbols can be annotated by a numerical superscript, that is called the *level* of the symbol. Symbols with superscripts are called annotated symbols. L^+ extends L with two infinite sets of constants: a set $\mathcal{A}$ containing a constant for each level $k \in \mathbb{N}$ and a set $\mathcal{C}$, containing an infinite number of constants for each level $k > 0$. Moreover, L^+ contains the annotated versions of the constants and functional symbols in L. Precisely:

Definition 1. Let L be a modal language, $\mathcal{A} = \{a_0^0, a_1^1, ...\}$ and $\mathcal{C}$ disjoint sets of annotated constants different from any symbol in L, such that $\mathcal{A}$ contains exactly one constant a_k^k for each $k \in \mathbb{N}$, and $\mathcal{C}$ contains a denumerable set of constants $c_{k_1}^k, c_{k_2}^k, ...$ annotated with level k, for each $k \geq 1$. Then the *labelled extension of* L is the language L^+ such that $L_P^+ = L_P$ and L_F^+ is:

$$L_F \cup \mathcal{C} \cup \mathcal{A} \cup \{f^k \mid f \in L_F \text{ and } k \in \mathbb{N}\}$$

In the definition above, the elements of $\mathcal{C}$ are assumed to be pairwise distinct, independently of their annotation (i.e. if c_k^n and c_k^m are in $\mathcal{C}$, then $n = m$).

The nodes of a ground tableau for a set of closed formulae in the modal language L (called the *original language*) are labelled sets of the form $n : S$ where n is a natural number, called the *depth* of the node, and S is a set of closed modal formulae in the language L^+, that will be called the *language* *of the tableau.* Similarly to the sequent calculus in [19], there is no labelling of single formulae, but of whole sets of formulae.

Symbol annotations and node depths are related: let t be the occurrence of a term in a node $n : S$, that we can assume describes a given world w; then the levels of the symbols in t tell us whether t has a denotation in the domain of w, and whether its denotation can be assumed to be the same as the denotation of another occurrence of the same term t (with possibly different annotations).

Next, we define the initial tableau for S and an operation that is used in some propositional expansion rules. From now on, we assume that a given logic has been fixed, with a propositional basis that is one of the analytic logics **K**, **D**, **K4**, **T**, or **S4**, and a first-order variant. q

DEFINITION 2.

1. If S is a set of modal formulae, then the initial tableau for S is the single node $1 : S^*$, where:

   local terms, rigid designation: S^* is obtained from S by annotating each functional symbol with level 0.

   local terms, non-rigid designation: S^* is obtained from S by annotating each *occurrence* of a functional symbol outside the scope of any modal operator with level 1.

   non-local terms: $S^* = S$.

2. If S is a set of modal formulae and $n \in \mathbb{N}$, then:

   local terms: S^n is obtained from S by annotating each non-annotated *occurrence* of a functional symbol outside the scope of any modal operator with level n;

   non-local terms: $S^n = S$.

The main intuition behind the notions above is that symbols annotated with level 0 are always symbols of the original language L (and have not been introduced by means of the δ-rule, defined below), their interpretation is the same in every possible world and the domain of every world is closed with respect to the application of the functions they denote. Thus, symbols of level 0 occur only in the calculus for the DDE variant with rigid designation and local terms.

In the case of non-rigid designation and local terms, on the contrary, a symbol of the original language L can occur with different levels in a tableau branch: when occurring with level n, it actually refers to the denotation of the symbol in the world corresponding to the node with depth n in that branch. Symbols occurring outside the scope of a modal operator are initialized with level 1, the depth of the initial node. Symbols of L_F occurring inside a modal operator receive their levels only when, by application of a propositional rule, they come to the surface (by means of operation 2 in Definition 2).

In the case of non local terms, on the contrary, no assumption is made on the denotation of symbols of the base language: they have no level in the initial node of the tableau and never get one.

We consider a simple set of propositional tableau rules (others may be chosen as well), shown in Figure 1, where S, S' are sets of modal formulae (in the language of the tableau), $\Box S$ stands for $\{\Box G \mid G \in S\}$, S' is a set of non-boxed modal formulae, comma is set union. The propositional part of the tableau systems we consider consists of the classical rules, and: the rule π_K in the systems **K**, **D** and **T**, the rule π_4 in **K4** and **S4**, the rule ν_T in **T** and **S4**, the rule ν_D in **D**.

Adopting a terminology from [15], rules that do not modify the depth of the node are called *static*, the others are called *dynamic*. So, a non-annotated symbol can obtain a superscript only with the application of dynamic rules.

As can be noted, the depth n of a node $n : S$ in a tableau "counts" the number of dynamic rules applied in the branch before $n : S$. And, in fact, node labels are not essential in the calculi and the depth of a node could be defined as a meta-level concept. For this reason the present approach is not in the style of prefixed tableaux.

The next definition establishes the relation between node depths and symbol levels, by means of the concept of *domain* of a node.

Definition 3.

1. A term t in L^+ is at **depth** n iff every symbol in t has a level and t contains only symbols with:

   varying domains, non-rigid designation: level n;

   varying domains, rigid designation: level n and level 0;

   cumulative domains: level $k \leq n$.

2. If n is natural number, then the *dummy constant of level* n is the constant $a_p^p \in \mathcal{A}$, where

   varying domains, rigid designation, local terms: if there is at least a constant in L then $p = 0$, otherwise $p = n$;

   varying domains, non-rigid designation $p = n$;

   varying domains, non-local terms $p = n$;

   cumulative domains: $p = 1$

Classical rules	
(α) $\dfrac{n : A \wedge B, S}{n : A, B, S}$	(β) $\dfrac{n : A \vee B, S}{n : A, S \qquad n : B, S}$
ν-rules	
(ν_D) $\dfrac{n : \Box S, S'}{n+1 : S^{n+1}}$	(ν_T) $\dfrac{n : \Box A, S}{n : A^n, \Box A, S}$
π-rules	
(π_K) $\dfrac{n : \Diamond A, \Box S, S'}{n+1 : A^{n+1}, S^{n+1}}$	(π_4) $\dfrac{n : \Diamond A, \Box S, S'}{n+1 : A^{n+1}, S^{n+1}, \Box S}$

Figure 1. Propositional expansion rules

3. The domain of a tableau node $n : S$ is:

$$\text{dom}(n : S) = \{a_p^p\} \cup \{t \mid t \text{ occurs in } S \text{ and } t \text{ is at depth } n\},$$

where a_p^p is the dummy constant of level n.

Roughly speaking, the object denoted by a term in $\text{dom}(n : S)$ can be assumed to belong to the domain of the world "corresponding" to the node $n : S$. Such terms are formed by use of symbols of level n itself, symbols of level 0 in the case of rigid designation (i.e. symbols of the base language L), and symbols having lower level in the case of cumulative domains, e.g. constants denoting objects belonging to the domain of some previous world, therefore also to the considered world.

Note that an annotated term may be at more than one depth. Moreover, if $t \in \text{dom}(n : S)$, then t is a ground term, all the symbols occurring in t are annotated and every sub-term t' of t is in $\text{dom}(n : S)$.

The quantifier expansion rules in Tab_G are the following:

(γ) $\dfrac{n : \forall x A, S}{n : A[t/x], \forall x A, S}$	(δ) $\dfrac{n : \exists x A, S}{n : A[c^n/x], S}$

In the γ-rule, t is any ground term in $\text{dom}(n : \forall x A, S)$. In the δ-rule, c is a constant in $\mathcal{C}$ that does not occur in $\{\exists x A\} \cup S$. Note that c needs not be new in the whole tableau: the local restriction on the δ-rule corresponds exactly to the *eigenvariable* condition in sequent calculi.

Since $\mathrm{dom}(n : \forall x A, S)$ is never empty, even if $n : \forall x A, S$ does not contain any ground term, the γ-rule can always be applied at least once to such a node. The dummy constants are in fact used to take into account the fact that the object domains are never empty, in much the same way as an additional constant is used to form Herbrand domains in classical logic when there are no ground terms. The difference, here, is that we may need to have an infinite number of such constants, one for each depth. If domains are cumulative, then one is enough (a_1^1): it denotes any object in the initial world w_0, that exists in any other world accessible from w_0. Also if domains can vary but the original language L contains at least a constant c and terms are assumed to be local, one additional constant (a_0^0) is enough: in fact, if terms of are local, c denotes an object existing in every world, and the constant a_0^0 will be taken to denote such an object. If none of the previous cases applies, then it may be the case that any two object domains have empty intersection, so we need a different constant to name an arbitrary object of each domain.

In Tab_G, once a symbol occurrence gets a level, it never changes. In particular, new constants introduced by the δ-rule never change their level and the only symbols that may occur with different annotations in a tableau node are symbols from the base language, when we are in the case of non-rigid designation and local terms. In that case, f^n is the denotation of f in the world corresponding to the node of depth n, that can be different from the denotation of the same symbol in another world. In consideration of this fact, the notion of complementary literals does not ignore the level of symbols. For example, $p(c^1)$ and $\neg p(c^1)$ are complementary, but $p(c^1)$ and $\neg p(c^2)$ are not.

DEFINITION 4. Two literals P and $\neg Q$ in the language L^+ of a tableau are complementary iff P and Q are identical, including their annotations. A node is closed if it contains two complementary literals. A tableau is closed iff all its leaves are closed.

Note that no functional symbol with level greater than n may occur in a node $n : S$ of a tableau. In the case of rigid designation and local terms, every occurrence of a functional symbol in a tableau is annotated; if designation is non-rigid, a functional symbol may occur with no level in a tableau only if it is in the scope of a modal operator, and no symbol is annotated with level 0. Moreover, if $n + 1 : S$ is obtained from $n : S'$ by application of a dynamic rule, then S can contain symbols with level $n+1$ only if designation is non-rigid and terms are local; such symbols occur without level in S', i.e. they are symbols of the original language L

In the case of cumulative domains, rigid designation and local terms, the calculus is not different from the tableau system described in [11], Chapter 7 (where, however, the language does not contain any functional symbol with non-null arity). In fact, every functional symbol in a tableau is annotated and $n : S$ never contains symbols of level greater than n. So every term occurring in $n : S$ is in $\mathrm{dom}(n : S)$.

Here follow some examples. The tableau on the left below proves the validity of **CBF** in the cumulative domain logics, using the π_K-rule. Since there are no functional symbols in L_F, there is no distinction to be made about the designation of terms. The last step in the varying domain systems would not be allowed: $\forall x p(x)$ can be instantiated only with terms at depth 2. Moreover, since there are no choice points above and the last expansion can only be replaced by an application of the γ-rule on the constant a_2^2 (there are no constants in L_F), **CBF** cannot be proved in the varying domain systems.

$$\dfrac{\dfrac{\dfrac{\dfrac{1 : \neg(\Box\forall x p(x) \to \forall x \Box p(x))}{1 : \Box\forall x p(x),\ \neg\forall x\Box p(x)}\,(\alpha)}{1 : \Box\forall x p(x),\ \neg\Box p(c^1)}\,(\delta)}{2 : \forall x p(x),\ \neg p(c^1)}\,(\pi_K)}{2 : p(c^1),\ \forall x p(x),\ \neg p(c^1)}\,(\gamma) \qquad \dfrac{\dfrac{\dfrac{\dfrac{1 : \neg(\forall x\Box p(x) \to \Box\forall x p(x))}{1 : \forall x\Box p(x),\ \neg\Box\forall x p(x)}\,(\alpha)}{1 : \Box p(a^1),\ \forall x\Box p(x),\ \neg\Box\forall x p(x)}\,(\gamma)}{2 : p(a^1),\ \neg\forall x p(x)}\,(\pi_k)}{2 : p(a^1),\ \neg p(c^2)}\,(\delta)$$

A similar reasoning shows that no instance of the Barcan formula (**BF)** $\forall x\Box A \to \Box\forall x A$ that characterizes constant domains, is provable. The tableau above on the right is an attempt, using again the π_K-rule and the instance where $A = p(x)$. In this case there is no difference between the cumulative and varying domain variants of the calculus. Since with the application of the π_K-rule the universal formula is necessarily lost, the tableau cannot be closed.

Below, on the left, there is a proof of $\Box(\forall x p(x) \to p(c))$ (in any propositional basis) with rigid designation (either varying or cumulative domains: in both cases the term c^0 is at any depth) and local terms, showing that, in this variant of the systems, the rule of instantiation is necessarily valid. The same formula has a different proof (on the right, below) in the case of non-rigid designation (and local terms).

$$\dfrac{\dfrac{\dfrac{1 : \neg\Box(\forall x p(x) \to p(c^0))}{2 : \neg(\forall x p(x) \to p(c^0))}\,(\pi)}{2 : \forall x p(x),\ \neg p(c^0)}\,(\alpha)}{2 : p(c^0),\ \forall x p(x),\ \neg p(c^0)}\,(\gamma) \qquad \dfrac{\dfrac{\dfrac{1 : \neg\Box(\forall x p(x) \to p(c))}{2 : \neg(\forall x p(x) \to p(c^2))}\,(\pi)}{2 : \forall x p(x),\ \neg p(c^2)}\,(\alpha)}{2 : p(c^2),\ \forall x p(x),\ \neg p(c^2)}\,(\gamma)$$

And it cannot be proved when terms are not necessarily local. Here follows a failed attempt:

$$\cfrac{\cfrac{\cfrac{1 : \neg\Box(\forall x p(x) \to p(c))}{2 : \neg(\forall x p(x) \to p(c))}\,(\pi)}{2 : \forall x p(x),\ \neg p(c)}\,(\alpha)}{2 : p(a_2^2),\ \forall x p(x),\ \neg p(c)}\,(\gamma)$$

The next example is a (failed) attempt to show that $p(c), \forall x(p(x) \to \Box p(x)) \vdash \Box p(c)$ in the case of non-rigid designation and local terms:

$$\cfrac{\cfrac{1 : \forall x(p(x) \to \Box p(x)),\ p(c^1),\ \neg\Box p(c)}{1 : \forall x(p(x) \to \Box p(x)),\ p(c^1) \to \Box p(c^1),\ p(c^1),\ \neg\Box p(c)}\,(\gamma)}{\begin{array}{c}1 : \forall x(p(x) \to \Box p(x)),\\ \neg p(c^1),\ p(c^1),\ \neg\Box p(c)\end{array} \qquad \cfrac{\begin{array}{c}1 : \forall x(p(x) \to \Box p(x)),\\ \Box p(c^1),\ p(c^1),\ \neg\Box p(c)\end{array}}{2 : p(c^1),\ \neg p(c^2)}\,(\pi_k)}\,(\beta)$$

The leftmost leaf is closed, but the rightmost is not, because $p(c^1)$ and $\neg p(c^2)$ are not complementary. And in fact $\Box p(c)$ is not a logical consequence of $p(c)$ and $\forall x(p(x) \to \Box p(x))$ in the case of non-rigid designation and local terms. If terms are not local, the very first inference is not correct. In fact, the tableau is initialized without any annotation on symbols and the γ-rule can never be applied to a term with no levels.

2.1. Soundness of Tab_G

In the rest of this Section we are going to prove that Tab_G is sound and complete. First of all we state a modal counterpart of the Substitution Theorem, that can easily be proved by induction on formulae:

THEOREM 1. *Let L be a language, $\mathcal{M} = \langle W, w_0, R, D, \delta, \phi, \pi, \rangle$ an interpretation of L, $w \in W$, A a formula with only the variable x free, and t, t' ground terms of L.*

1. *If the interpretation of t and t' is rigid and equal, i.e. for all $w' \in W$, $\phi(w', t) = \phi(w', t') = d$ for some fixed $d \in D$, then $\mathcal{M}, w \models A[t/x]$ iff $\mathcal{M}, w \models A[t'/x]$.*
2. *If x does not occur in A in the scope of any modal operator and $\phi(w, t) = \phi(w, t')$, then $\mathcal{M}, w \models A[t/x]$ iff $\mathcal{M}, w \models A[t'/x]$.*

Next, we introduce the notion of satisfiability of tableau nodes. Here, when dealing with interpretations of annotated terms we intend that annotations are not ignored and actually make symbols different. For instance, if f^k is a symbol annotated with level k, then $\phi(w, f^k)$ may be different from $\phi(w, f)$, or $\phi(w, f^j)$ when $j \neq k$. Moreover, we assume that the interpretation of an annotated symbol is always rigid. Obviously, we can have $\phi(w, f^k) = \phi(w', f^j)$, when $j \neq k$.

DEFINITION 5.

1. An interpretation of an annotated language L^+ is a modal interpretation $\mathcal{M} = \langle W, w_0, R, D, \delta, \phi, \pi, \rangle$, where ϕ gives an interpretation to annotated and non-annotated symbols, in such a way that symbols with different annotations f^k and f^j are considered as different symbols, and both different from f. Moreover, for all annotated symbol f^k and for all $w, w' \in W$, $\phi(w, f^k) = \phi(w', f^k)$
2. Let $\mathcal{M}$ be an interpretation and w a world in $\mathcal{M}$. Then $\mathcal{M}, w \models n : S$ iff:
   - $\mathcal{M}, w \models S$ and
   - for every ground term $t \in \mathrm{dom}(n : S)$, $\mathcal{M}(w, t) \in \delta(w)$.
3. A node $n : S$ is satisfiable if there exists an interpretation $\mathcal{M}$ and a world w in $\mathcal{M}$ such that $\mathcal{M}, w \models n : S$.

Modulo the definition above, the soundness proof runs along standard lines.

THEOREM 2 (Soundness of Tab_G). *If there exists a closed tableau for a set of modal formulae S, then S is unsatisfiable.*

PROOF. First of all, observe that if there exists a closed tableau for S, then there is also a closed tableau where every application of the δ-rule in a branch chooses a constant (a δ-constant) that is new *in the branch*. Therefore, it is sufficient to prove that if S is satisfiable, then any tableau for S, where δ-constants are new in the branch, is open.

Let $\mathcal{T}$ be any of such tableaux for S, and L the language of S. We prove that $\mathcal{T}$ is open, by showing that there exists at least a leaf $m : S'$ in $\mathcal{T}$, an interpretation $\mathcal{M}$ and a world w in $\mathcal{M}$ such that $\mathcal{M}, w \models m : S'$. Obviously, if $m : S'$ is satisfiable, S' does not contain complementary literals, hence it is open.

Below, we inductively define an interpretation $\mathcal{M}' = \langle W, w_0, R, D, \delta, \phi', \pi\rangle$, of symbols from L^+ starting from the model $\mathcal{M} = \langle W, w_0, R, D, \delta, \phi, \pi\rangle$, of the initial set of formulae S (in the language L), given by hypothesis. The two interpretations differ only because the interpretation function ϕ' extends ϕ to (a subset of) L^+. The interpretation of symbols with annotation in $\mathcal{M}'$ is always rigid and, since the constants $a_i \in \mathcal{A}$ do not occur in S, we assume that their interpretations are such that for all $w \in W$ and $n \in \mathbb{N}$, $\phi'(w, a_n^n)$ is any element in the domain $\delta(w)$, and, if we are in the case of varying domains, rigid designation and local terms, and S contains at least a constant c, then $\phi'(w, a_0^0) = \phi(w, c)$.

Base: if $\mathcal{T}$ is the initial tableau for S, then it consists of the leaf $1 : S^*$. Let $\mathcal{M} = \langle W, w_0, R, D, \delta, \phi, \pi \rangle$ be a model of S, given by hypothesis. We consider the different ways of building S^*:

local terms, rigid designation: every functional symbol in S^* is annotated with level 0. Then ϕ' extends ϕ in such a way that for all $w \in W$ and symbol f:

$$\phi'(w, f^0) = \phi(w, f)$$

Since the interpretation of terms is rigid, by the first item in Theorem 1, $\mathcal{M}', w_0 \models S^*$.

local terms, non-rigid designation: in S^*, only symbols occurring outside the scope of any modal operator get an annotation, namely level 1. In this case we establish that for all $w \in W$

$$\phi'(w, f^1) = \phi(w_0, f)$$

By the second item in Theorem 1, $\mathcal{M}', w_0 \models S^*$.

non-local terms: since $S^* = S$, we just set $\mathcal{M}' = \mathcal{M}$.

Now, it remains to be proved that for all ground terms $t \in \mathrm{dom}(1 : S^*)$, $\phi'(w_0, t) \in \delta(w_0)$. The dummy constant in $\mathrm{dom}(1 : S)$ is either a_0^0 or a_1^1, and both their interpretations belong to $\delta(w_0)$, by the above assumptions on the interpretation of constants in $\mathcal{A}$. In the case of non-local terms, there are no other elements in $\mathrm{dom}(1 : S^*)$. Otherwise, the other elements in $\mathrm{dom}(1 : S^*)$ are ground terms occurring in S, whose interpretations belong to $\delta(w_0)$, since terms are local.

Induction Step: we distinguish several cases, according to the last expansion rule applied to a satisfiable leaf in $\mathcal{T}$, and show that, if

$$\frac{n : S}{m : S_0} \quad \text{or} \quad \frac{n : S}{m : S_0 \quad m : S_1}$$

is an instance of an expansion rule, and $n : S$ is satisfiable then [either] $m : S_0$ [or $m : S_1$] is satisfiable. When considering models $\mathcal{M}$ of a node $n : S$, we assume that symbols in $\mathcal{C}$ that do not occur in S are not interpreted (we are under the assumption that δ-constants are new in the branch). We show here the treatment of the quantifier rules and a dynamic rule, as sample cases.

γ-rule: $\dfrac{n : \forall x A, S}{n : A[t/x], \forall x A, S}$ where t is any ground term belonging to $\mathrm{dom}(n : \forall x A, S)$. Let us assume that $\mathcal{M}, w \models n : \forall x A, S$, hence in particular, for all $d \in \delta(w)$ $\mathcal{M}, w \models A[\overline{d}/x]$. From Definition 5 it follows also that, if $t \in \mathrm{dom}(n : \forall x A, S)$ and $\mathcal{M}(w, t) = d$, then $d \in \delta(w)$. Moreover, since t only contains annotated symbols, its interpretation is rigid. The interpretation of $\overline{d}$ is rigid too, so that, by item 1 in Theorem 1, $\mathcal{M}, w \models A[t/x]$.

Finally, let us assume that $t \in \mathrm{dom}(n : A[t/x], \forall x A, S)$. We prove $\phi(w, t) \in \delta(w)$ by induction on t: if t is a constant, then it occurs also in $\forall x A, S$, so $t \in \mathrm{dom}(n : \forall x A, S)$ and $\phi(w, t) \in \delta(w)$ follows from the inductive hypothesis. Otherwise, let $t = f^k(t_1, ..., t_m)$, with $m > 0$, i.e. f a symbol of the original language L, and $\phi(w, t_i) \in \delta(w)$, for $1 \leq i \leq m$. Since f has a level, terms are local, i.e. $\delta(w)$ is closed w.r.t. the application of $\phi(w, f)$. Therefore $\phi(w, t) \in \delta(w)$.

δ-rule: $\dfrac{n : \exists x A, S}{n : A[c^n/x], S}$. Let us assume that $\mathcal{M}, w \models n : \exists x A, S$ and that $\mathcal{M}$ assigns no meaning to c^n (that is new). Then we can extend the interpretation, like in the corresponding proof in classical logic, so that c^n (rigidly) denotes any element d of $\delta(w)$ satisfying A: i.e. choose $d \in \delta(w)$ such that $\mathcal{M}, w \models A[\overline{d}/x]$ and set $\mathcal{M}'(w^*, c^n) = d$ *for all worlds* $w^* \in W$. Then, obviously $\mathcal{M}'(w, c^n) \in \delta(w)$. With this observation, $\mathcal{M}'(w, t) \in \delta(w)$, for all $t \in \mathrm{dom}(n : A[c^n/x], S)$ is proved like in the previous case.

Finally, $\mathcal{M}', w \models A[c^n/x]$ follows from $\mathcal{M}, w \models A[\overline{d}/x]$ and the fact that c^n rigidly denotes d (by the first item in Theorem 1).

ν_D-rule: $\dfrac{n : \Box S, S'}{n + 1 : S^{n+1}}$. If $\mathcal{M}, w \models n : \Box S, S'$ and the frame is serial, then there exists w' such that wRw', and clearly $\mathcal{M}, w' \models S$. If we are in the case of non-local terms or rigid designation, then $S^{n+1} = S$ (if designation is rigid, all symbols get the annotation 0 in the initial tableau, and there are no non-annotated symbols in S), so that trivially $\mathcal{M}, w' \models S^{n+1}$. If terms are local and designation non-rigid, we extend $\mathcal{M}$ to $\mathcal{M}'$ by setting $\phi'(w^*, f^{n+1}) = \phi(w', f)$ for all $w^* \in W$ and f^{n+1} occurring in S^{n+1}. Since the occurrences of symbols annotated with $n + 1$ are not in the scope of any modal operator in S^{n+1}, $\mathcal{M}', w' \models S^{n+1}$ follows from item 2 of Theorem 1. It remains to be shown that for all terms $t \in \mathrm{dom}(n + 1 : S^{n+1})$, $\phi'(w', t) \in \delta(w')$. The proof is by induction on t.

- If $t = a_p^p$ is the dummy constant of level $n+1$, or any other element of $\mathcal{A}$, then $\phi'(w', t) \in \delta(w')$ follows from the assumption on the interpretation of dummy constants.
- If $t = f^0(t_1, ..., t_m)$, for $t_1, \ldots, t_m \in \text{dom}(n+1 : S^{n+1})$, then we are in the case of local terms and rigid designation (there are no symbols of level 0 otherwise) and f is a symbol in the original language L. By the induction hypothesis for all $i = 1, \ldots, m$, $\phi'(w', t_i) \in \delta(w')$, so that $\phi'(w', t) \in \delta(w')$ by the assumption of locality of term.
- If $t = f^k(t_1, ..., t_m)$, for $t_1, \ldots, t_m \in \text{dom}(n+1 : S^{n+1})$ and $k > 0$, then for all $i = 1, \ldots, m$, $\phi'(w', t_i) \in \delta(w')$ (by the induction hypothesis), and:

  varying domains: it must be $k = n+1$. Then f is a symbol in the original language L and we are in the case of non-rigid designation and local terms, so that $\phi'(w', f^{n+1}) = \phi(w', f)$. So, $\phi'(w', t) \in \delta(w')$ follows from the inductive hypothesis and the assumption of locality of term.

  cumulative domains: it must be $k \leq n+1$. If $k = n+1$, then, like in the case above, f is a symbol in the language L (and we are in the case of non-rigid designation and local terms), so that $\phi'(w', t) \in \delta(w')$ follows as before. Otherwise t occurs in $\Box S$; by the inductive hypothesis, $\phi'(w, t) \in \delta(w)$. Since $\delta(w) \subseteq \delta(w')$, $\phi'(w, t) \in \delta(w')$. Moreover, since t only contains annotated symbols, and their interpretation is rigid, $\phi'(w, t) = \phi'(w', t)$, i.e. $\phi'(w', t) \in \delta(w')$. ■

2.2. Completeness of $\boldsymbol{Tab_G}$

The completeness proof goes along the following lines. If L is a modal language, we define its *relevant extension* L^* as the sublanguage of L^+ such that L^*_F contains exactly: the elements of $\mathcal{C}$ (we recall that each constant in $\mathcal{C}$ is annotated by a single level), the dummy constants of level n, for each $n \geq 1$ (in dependence of the DDE-variant of QML), and the following functional symbols from L^+_F:

local terms, rigid designation: f^0 for all $f \in L_F$;

local terms, non-rigid designation: f and f^k for all $f \in L_F$ and $k \geq 1$;

non-local terms: f for $f \in L_F$ (with no annotation);

In other words, if S is a set of formulae in the language L, L^* contains symbols that can actually be used in a tableau for S.

We first-prove the completeness of a tableau systems where the γ-rule is replaced by:

$$(\gamma^*) \quad \frac{n : \forall x A, S}{n : A[t/x], \forall x A, S} \qquad \text{where } t \text{ is any ground term at depth } n \text{ of the language } L^*$$

Here, we assume that L is the language of the initial set of formulae and L^* its relevant extension. A tableau in this calculus will be called a γ^*-tableau. Completeness is proved by construction of a model of any set of sentences that has no closed γ^*-tableau, and the construction obviously exploits the "wasteful" γ^*-rule. Therefore, if a set S of formulae is unsatisfiable, then there is a closed γ^*-tableau for S. Then we show that any γ^*-expansion in a closed γ^*-tableau can be replaced by a γ-expansion, thus obtaining a closed tableau in Tab_G.

So we set to prove that if there exists no closed γ^*-tableau for a set of modal formulae S, then S is satisfiable. In the construction of the model of S we shall deal with labelled sets $n : S$ where S can be infinite. Such structures will be called "nodes", as well. Here follow some preliminary definitions and results.

Definition 6.

1. If S is a (finite or infinite) set of formulae, then S is consistent iff S does not contain any pair of complementary literals.
2. Let $n > 0$ be a natural number, and S a set of modal formulae in L^*. Then $n : S$ is downward saturated iff S is consistent and $n : S$ is closed with respect to the application of static rules, i.e.
   a) S is consistent;
   b) if $A \wedge B \in S$ then $A \in S$ and $B \in S$
   c) if $A \vee B \in S$ then either $A \in S$ or $B \in S$;
   d) if $\forall x A \in S$ then $A[t/x] \in S$ for every ground term t at depth n in L^*;
   e) if $\exists x A \in S$ then $A[c^n/x] \in S$ for some constant c^n;
   f) if the logic is either **T** or **S4** and $\Box A \in S$ then $A^n \in S$.
3. Let S be a (finite or infinite) set of formulae and $n > 0$ a natural number. $n : S$ is tab-consistent iff for every finite subset $S' \subseteq S$ there is no closed γ^*-tableau for $n : S'$.

Lemma 1. *If $n : S$ is tab-consistent, then there exists a set $S^\infty \supseteq S$ such that $n : S^\infty$ is downward saturated.*

Proof. The node $n : S^\infty$ is the closure of $n : S$ with respect to the application of static expansion rules. Precisely, assuming that any fair strategy for the application of static rules is given, we inductively define a sequence $n : S_0$, $n : S_1$, $\ldots$ of tab-consistent nodes as follows:

1. $n : S_0 \;=\; n : S$.

2. Let $\dfrac{n : S_k}{n : S' \qquad [n : S'']}$ be the instance of the static expansion rule applicable to $n : S_k$, given by the rule application strategy. Since $n : S_k$ is tab-consistent, at least one of $n : S'$ and $n : S''$ is tab-consistent. Then $S_{k+1} = S'$ if $n : S'$ is tab-consistent, $S_{k+1} = S''$ otherwise.

The set S^∞ is the union of the sets S_k:

$$S^\infty = \bigcup_{k=0}^{\infty} S_k$$

hence, in particular $S \subseteq S^\infty$.

Since we are assuming to follow a fair construction strategy, the labelled set $n : S^\infty$ obviously satisfies conditions (b)-(f) of Definition 6.2. Let us assume that for some atomic formula A, both $A \in S^\infty$ and $\neg A \in S^\infty$. Then A and $\neg A$ belong to some S_k, since literals are never deleted by any static expansion rule. But S_k is tab-consistent by construction, therefore $n : S^\infty$ satisfies also condition (a) of Definition 6.2. ■

In the following, we use the operation $\#$ on sets of formulae defined as follows: $S\#$ is $\{A \mid \Box A \in S\}$ if the propositional basis of the logic is either **K**, **D** or **T**, otherwise (**K4** or **S4**) $S\#$ is $\{A, \Box A \mid \Box A \in S\}$.

LEMMA 2. *If $n : S$ is tab-consistent, then for any formula $\Diamond A$ in S, the labelled set $n+1 : A^{n+1}, S\#^{n+1}$ is tab-consistent. Moreover, if the logic is **D**, then $n+1 : S\#^{n+1}$ is tab-consistent.*

PROOF. Let us assume that there is a closed tableau for some finite subset of $n+1 : A^{n+1}, S\#^{n+1}$. The initial node X of such a tableau has either the form

$$\{n+1 : A^{n+1}, B_0^{n+1}, B_1^{n+1}, ..., B_k^{n+1}\} \text{ or } \{n+1 : B_0^{n+1}, B_1^{n+1}, ..., B_k^{n+1}\}$$

for some $B_0, B_1, \ldots, B_k$ belonging to $S\#$. Now, if there is a closed tableau for $\{n+1 : B_0^{n+1}, B_1^{n+1}, \ldots, B_k^{n+1}\}$, then there is also a closed tableau for $\{n+1 : A^{n+1}, B_0^{n+1}, B_1^{n+1}, \ldots, B_k^{n+1}\}$, so we may assume that X actually contains A^{n+1}.

For each $i = 0, \ldots, k$, either $\Box B_i \in S$ or (if the logic is either **S4** or **K4**) $B_i \in S$ (and $B_i^{n+1} = B_i$). Let S' be the finite subset of S containing such "ancestors" of the B_i^{n+1}'s and $\Diamond A$. Then X can be obtained from $n : S'$ by application of the π rule of the logic, so there exists a closed tableau for $n : S'$, and $n : S$ is not tab-consistent.

Now, let us assume that the logic is **D**. If there is a finite subset S' of $\{B^{n+1} \mid \Box B \in S\}$ such that there is a closed tableau for $n+1 : S'$, then an application of the ν_D rule on top makes it a closed tableau for some finite subset of $n : S$. ∎

The next step consists in the construction of the **syntactical interpretation** of S, for any S such that $1 : S^*$ is tab-consistent. The construction follows the standard approach. The frame $\mathcal{F} = \langle W, w_0, R\rangle$ of the interpretation is defined as follows:

1. The elements of W are sets of modal formulae in L^*. A natural number depth(w) is associated to each $w \in W$, in such a way that depth(w) : w is tab-consistent and downward saturated. The set W is built by steps. The same induction defines the function *depth* : $W \to \mathbb{N}$ and a binary relation $\prec$ on W, and shows that each labelled set depth(w) : w, for $w \in W$, is tab-consistent and downward saturated.

   Step 1. Let w_0 be any superset of S^* such that $1 : w_0$ is downward saturated. Such a set exists by Lemma 1, because $1 : S^*$ is tab-consistent. Then $w_0 \in W$ and depth(w_0) = 1. Obviously, $1 : w_0$ is tab-consistent and downward saturated.

   Step k+1 ($k > 0$). For each w such that depth(w) $= k$ and for every formula $\Diamond A$ in w: let w' be any superset of $\{A^{k+1}\} \cup w\#^{k+1}$ such that $k{+}1 : w'$ is downward saturated. Such a set exists because, since $k : w$ is tab-consistent, then, by Lemma 2, $k+1 : \{A^{k+1}\} \cup w\#^{k+1}$ is tab-consistent, and Lemma 1 can be applied. Then $w' \in W$ and depth(w') $= k+1$. Moreover, the pair $\langle w, w'\rangle$ is added to $\prec$ (i.e. we set $w \prec w'$).

   If the logic is **D**, then also choose any superset w' of $w\#^{k+1}$ such that $k+1 : w'$ is downward saturated. Such a set exists because, since $k : w$ is tab-consistent, then, by Lemma 2, $k+1 : w\#^{k+1}$ is tab-consistent, and Lemma 1 can be applied. Then $w' \in W$ and depth(w') $= k+1$. Moreover, $w \prec w'$.
2. R is $\prec$ if the logic is based on either **K** or **D**, the reflexive closure of $\prec$ for the logic **T**, the transitive closure of $\prec$ for **K4**, and the reflexive and transitive closure of $\prec$ in the case of **S4**.

The frame $\mathcal{F}$ is clearly a frame of the considered logic. The syntactical interpretation $\mathcal{M} = \langle W, w_0, R, D, \delta, \phi, \pi\rangle$ of S is an interpretation of the language L^*, whose frame is $\mathcal{F} = \langle W, w_0, R\rangle$, and:

1. D is the set of ground terms in L^*, possibly including terms that are not "well formed", for instance $f^1(c^2)$; however, such elements belong to no world domain (see next point).

2. For every $w \in W$, with $\text{depth}(w) = n$, $\delta(w)$ is the set of ground terms of L^* at depth n.
3. For all $w \in W$ and functional symbol f^m annotated with level m in L^*, for all $t_1, ..., t_k \in D$:
$$\phi(w, f^m)(t_1, ..., t_k) = f^m(t_1, ..., t_k)$$
Moreover, if f is not annotated, then

    local terms: $\phi(w, f) = \phi(w, f^n)$ where $n = \text{depth}(w)$
    non-local terms: $\phi(w, f)(t_1, ..., t_k) = f(t_1, ..., t_k)$
4. For all $w \in W$ and k-ary predicate symbol p in L^*:
$$\pi(w, p) = \{\langle t_1, ..., t_k\rangle \mid p(t_1, ..., t_k) \in w\}$$

We remark that: (i) if $t \in \delta(w)$ for some $w \in W$, then every symbol in t has a level; (ii) the interpretation of symbols with levels is rigid; (iii) for all $w \in W$ and ground term t in L^*, if every symbol in t has a level, then $\phi(w, t) = t$; in particular, if $t \in \delta(w)$ then $\phi(w, t) = t$.

Now we prove some important properties of the syntactical interpretation of S. In the following, we assume that there exists no closed γ^*-tableau for S, and that $\mathcal{M} = \langle W, w_0, R, D, \delta, \phi, \pi\rangle$ is the syntactical interpretation of S.

LEMMA 3. *For any $w \in W$ and formula A: $\mathcal{M}, w \models \forall x A$ iff for all terms $t \in \delta(w)$, $\mathcal{M}, w \models A[t/x]$.*

PROOF. From the remarks above, if $t \in \delta(w)$ then $\phi(w, t) = t$ and t contains only symbols with a level, so that the interpretation of t is rigid. The result then follows from the first-item in Theorem 1:

$\mathcal{M}, w \models \forall x A$ iff for all $d \in \delta(w)$, $\mathcal{M}, w \models A[\overline{d}/x]$
iff for all $t \in \delta(w)$, $\mathcal{M}, w \models A[t/x]$ (Theorem 1) ■

LEMMA 4. *If terms are local, $w \in W$ with $\text{depth}(w) = n$, A is a formula in L^*, x a variable that occurs in the scope of no modal operator in A , t is a ground term in the language L^*, and t^n is the term obtained from t by assigning level n to all symbols without levels in t, then:*
$$\mathcal{M}, w \models A[t/x] \textit{ iff } \mathcal{M}, w \models A[t^n/x]$$

PROOF. If terms are local, then L^* contains symbols without level only if designation is non-rigid. In that case, with local terms, $\phi(w, f) = \phi(w, f^n)$. A straightforward induction on terms proves that $\phi(w, t) = \phi(w, t^n)$, so that Theorem 1 can be applied. ■

LEMMA 5. *For all $A \in w_0$, $\mathcal{M}, w_0 \models A$.*

PROOF. We show that, for all $w \in W$ and for every formula A: if $A \in w$ then $\mathcal{M}, w \models A$. The proof is by induction on A (remember that negation over non atomic formulae is a defined symbol). Let $w \in W$ and $n = \text{depth}(w)$.

1. A is a literal. By definition of π: $p(t_1, ..., t_k) \in w$ iff $\mathcal{M}, w \models p(t_1, ..., t_k)$. The result is then immediate if $A = p(t_1, ..., t_k)$.

   If $A = \neg p(t_1, ..., t_k)$ and $\neg p(t_1, ..., t_k) \in w$, then $p(t_1, ..., t_k) \notin w$, since w is consistent. Therefore $\mathcal{M}, w \models \neg p(t_1, ..., t_k)$.

2. $A = \forall x B$. If $\forall x B \in w$ then, since $n : w$ is downward saturated, $B[t/x] \in w$ for all ground terms t of L^* at depth n, i.e. for all $t \in \delta(w)$. By the induction hypothesis, for all $t \in \delta(w)$, $\mathcal{M}, w \models B[t/x]$. Therefore, $\mathcal{M}, w \models \forall x B$, by Lemma 3.

3. $A = \exists x B$. If $\exists x B \in w$, then for some constant c^n at depth n, $B[c^n/x] \in w$. By the induction hypothesis, $\mathcal{M}, w \models B[c^n/x]$ and since $c^n \in \delta(w)$, $\mathcal{M}, w \models \exists x B$.

4. $A = \Box B$. If $\Box B \in w$ and $\text{depth}(w) = n$, then for every w' such that wRw', $B^{n+k} \in w'$ for some $k \geq 0$. This is straightforward for the logics K and D. If R is reflexive (T and S4), then condition (f) in Definition 6.2 ensures that $B^n \in w$. If the logic is transitive (K4 and S4), then both B^{n+1} and $\Box B$ are in all the w' such that $w \prec w'$, so that B^{n+k} belongs to every w' such that wRw'.

   By the inductive hypothesis, if wRw' then $\mathcal{M}, w' \models B^{n+k}$, for some k such that $n + k = \text{depth}(w')$. In the case of non-local terms $B^{n+k} = B$, so that $\mathcal{M}, w' \models B$. Otherwise, by Lemma 4, $\mathcal{M}, w' \models B$. In both cases then $\mathcal{M}, w \models \Box B$.

5. $A = \Diamond B$. If $\Diamond B \in w$, then at step $n+1$ in the construction of W a world w' has been created such that $B^{n+1} \in w'$ and wRw'. By the inductive hypothesis, $\mathcal{M}, w' \models B^{n+1}$ and, using Lemma 4 like in the previous case, $\mathcal{M}, w' \models B$, so that $\mathcal{M}, w \models \Diamond B$. ∎

The next result proves that the γ^*-calculus is complete.

LEMMA 6. *If S is a set of unsatisfiable modal formulae, then there exists a closed γ^*-tableau for S.*

PROOF. We prove that if there exists no closed γ^*-tableau for S, then S is satisfiable. Let $\mathcal{M} = \langle W, w_0, R, D, \delta, \phi, \pi \rangle$ be the syntactical interpretation of S, L^* the relevant extension of L and L^- the version without levels of L^*, i.e. such that:

$$L_F^- = \{f \mid f^k \in L_F^* \text{ for some } k\}$$

The interpretation $\mathcal{M}'$ of L^- is the same as $\mathcal{M}$ except for the function ϕ, i.e. $\mathcal{M}' = \langle W, w_0, R, D, \delta, \phi', \pi \rangle$, where ϕ' is defined as follows:

1. if f is a symbol in the language L of S then for all $w \in W$:

   rigid designation, local terms: $\phi'(w, f) = \phi(w, f^0)$

   non-rigid designation, local terms: $\phi'(w, f) = \phi(w, f^n)$ where $n = \text{depth}(w)$

   non-local terms: $\phi'(w, f) = \phi(w, f)$.

2. if f is not a symbol in the language L, i.e. $f \in \mathcal{C} \cup \mathcal{A}$, then there is a single k such that $f^k \in L^*$; then $\phi'(w, f) = \phi(w, f^k)$.

Note that, obviously, in the local terms variant, terms are in fact local, by the construction of $\delta(w)$ and, in the case of rigid designation, the interpretation of terms in L^- is rigid. Finally, in the case of cumulative domains the corresponding restriction on domains is satisfied, because terms at depth n are also at depth m, when $m > n$.

If $t = f(t_1, ..., t_k)$ is a term with no annotations and $w \in W$, we define t^w inductively as follows: if f does not occur in the original language L and occurs with (the unique) level k in L^*, then $t^w = f^k(t_1^w, ..., t_k^w)$. Otherwise

rigid designation, local terms: $t^w = f^0(t_1^w, ..., t_k^w)$

non-rigid designation, non-local terms: $t^w = f^n(t_1^w, ..., t_k^w)$ where $n = \text{depth}(w)$

non-local terms: $t^w = f(t_1^w, ..., t_k^w)$

It is easy to show by induction on terms that for all $w \in W$ and term t in L^-, $\phi'(w, t) = t^w$. Vice-versa, if $w \in W$ and t is a term in L^* such that, in the case of non-rigid designation and local terms, every symbol f of the original language L has level $n = \text{depth}(w)$, then $t = t^w = \phi'(w, t^-)$, where t^- is obtained from t deleting all levels.

Now, let $w \in W$, A a formula with levels in the language L^*, such that, in the case of non-rigid designation and local terms, every symbol f of the original language L occurring in A outside the scope of a modal operator has level $n = \text{depth}(w)$. We prove that, if A^- denotes the formula obtained from A omitting all the levels, then

$$\text{if } \mathcal{M}, w \models A \text{ then } \mathcal{M}', w \models A^-$$

by induction on A. Let $n = \text{depth}(w)$.

1. A is an atom $p(t_1, ..., t_m)$ or the negation of an atom. Let us denote the version without levels of a term t_i by t_i^-. Then:

$$\begin{aligned}\mathcal{M}', w \models p(t_1^-, ..., t_m^-) &\text{ iff } \langle \phi'(w, t_1^-), ..., \phi'(w, t_m^-)\rangle \in \pi(w, p)\\ &\text{ iff } \langle (t_1^-)^w, ..., (t_m^-)^w\rangle \in \pi(w, p)\\ &\text{ iff } \langle t_1, ..., t_m\rangle \in \pi(w, p)\\ &\text{ iff } \mathcal{M}, w \models p(t_1, ..., t_m)\end{aligned}$$

2. $A = \forall x B$. If $\mathcal{M}, w \models A$ then for all $d \in \delta(w)$, $\mathcal{M}, w \models B[\overline{d}/x]$ and, by the induction hypothesis, $\mathcal{M}', w \models (B[\overline{d}/x])^-$. Since $(B[\overline{d}/x])^-$ is the same as $B^-[\overline{d}/x]$, it follows that $\mathcal{M}', w \models A^-$.
3. $A = \Diamond B$. If $\mathcal{M}, w \models A$ then there exists w' such that $w \prec w'$ and $\mathcal{M}, w' \models B^{n+1}$. By the inductive hypothesis $\mathcal{M}', w' \models (B^{n+1})^-$, i.e. $\mathcal{M}', w' \models B^-$, so $\mathcal{M}', w \models A^-$.

Finally, by Lemma 5, $\mathcal{M}, w_0 \models w_0$ and, since $S^* \subseteq w_0$, where $1 : S^*$ is the initial tableau for S, $\mathcal{M}, w_0 \models S^*$. Therefore, $\mathcal{M}', w_0 \models S$ because $(S^*)^- = S$. ■

The last step consists in showing that the γ^*-rule can be replaced by the γ-rule, without losing completeness. Such a result will also be used in Section 3.1.

LEMMA 7. *If there exists a closed γ^*-tableau for S, then there is a closed Tab_G tableau for S.*

PROOF. Let $\mathcal{T}$ be a closed γ^*-tableau for S. The proof is by induction on the number of applications of the γ^*-rule that are not instances of the γ-rule. If there are none, there is nothing to prove. Otherwise, let $\mathcal{B}$ be a branch in $\mathcal{T}$ containing one of such applications of the γ^*-rule and let:

$$(\mathcal{J})\ \frac{n : \forall x A, S}{n : A[t/x], \forall x A, S}$$

be the last application of such a rule. So, every symbol in t has a level, t is not the dummy constant of level n, and does not occur in $\forall x A, S$.

We then transform $\mathcal{T}$ by replacement of every occurrence of t below the expansion $\mathcal{J}$ by the dummy constant a_p^p of level n. The expansion $\mathcal{J}$ is a correct application of the γ-rule, since $a_p^p \in \mathrm{dom}(n : \forall x A, S)$.

Let $\mathcal{T}'$ be the subtree of $\mathcal{T}$ having $n : A[t/x], \forall x A, S$ as its root node (the expansion produced by $\mathcal{J}$). We show that the tree obtained from $\mathcal{T}'$ by replacing everywhere t with a_p^p is a Tab_G tableau and it is closed.

More generally, we prove that if $\mathcal{T}'$ is a closed tableau for $n : S[t/x]$ where t is a term at depth n, and if $\mathcal{T}'$ contains no application of the γ^*-rule, then $\mathcal{T}'$ can be transformed into a closed tableau $\mathcal{T}''$ for $n : S[a_p^p/x]$, just by replacing everywhere t with a_p^p. The proof is by induction on $\mathcal{T}'$. Below, we show the treatment of the base case and some cases in the induction step.

1. $\mathcal{T}'$ is a closed leaf: $m : B[t/x], \neg B[t/x], S[t/x]$, then obviously also $m : B[a_p^p/x], \neg B[a_p^p/x], S[a_p^p/x]$ is a closed leaf.

2. The first expansion in $\mathcal{T}'$ is an application of the γ-rule, having $t'[t/x]$ as its instance term (we emphasize that t' can contain t):

$$(\mathcal{J}') \ \frac{m : \forall y B[t/x], S[t/x]}{m : B[t/x][t'[t/x]/y], \forall y B[t/x], S[t/x]}$$

This application of the rule is transformed into:

$$(\mathcal{J}'') \ \frac{m : \forall y B[a_p^p/x], S[a_p^p/x]}{m : B[a_p^p/x][t'[a_p^p/x]/y], \forall y B[a_p^p/x], S[a_p^p/x]}$$

Now, if $t'[t/x]$ occurs in $\forall y B[t/x], S[t/x]$, then obviously $t'[a_p^p/x]$ occurs in $\forall y B[a_p^p/x], S[a_p^p/x]$. In order to show that this new expansion is a correct application of the γ-rule, we have to show that $t'[a_p^p/x]$ is at depth m. If t' does not contain x, there is nothing to prove ($t'[t/x] = t'[a_p^p/x]$). Otherwise:

cumulative domains: $p = 1$, so obviously if $t'[t/x]$ is at depth m with $m \geq n$ then also $t'[a_1^1/x]$ is at depth m.

varying domains: Note that in the varying domains case, a term at depth m can contain only symbols of level m and (if designation is rigid) 0. If $p = 0$ then we are in the case of rigid designation and local terms, and $t'[a_0^0/x]$ is at depth m.

If $p = n$ then t contains some symbol annotated with level n (if it contains only symbols with level 0 then there exists a constant of level 0 and terms are local: so we are in the case of rigid designation — there are no symbols with level 0 in either non-local terms or non-rigid designation tableau — and there is at least a constant in the base language L; therefore $p = 0$). Then it must be $n = m$, otherwise $t'[t/x]$ is not at depth m, so that also $t'[a_n^n/x]$ is at depth n.

Since the inference $\mathcal{J}''$ is a correct application of the γ-rule, the result follows from the inductive hypothesis.

3. The first expansion in $\mathcal{T}'$ is an application of the π_K-rule:

$$\frac{n : \Diamond A[t/x], \Box S[t/x], S'[t/x]}{n+1 : A[t/x]^{n+1}, S[t/x]^{n+1}}$$

Note that since t has only symbols with levels, it is not affected by the possible new labelling of symbols coming out of modal operators. So the expansion is transformed into:

$$\frac{n : \Diamond A[a_p^p/x], \Box S[a_p^p/x], S'[a_p^p/x]}{n+1 : A[a_p^p/x]^{n+1}, S[a_p^p/x]^{n+1}}$$

that is correct and the result follows by the inductive hypothesis. ∎

Lemmas 6 and 7 immediately imply the following

THEOREM 3 (Completeness of Tab_G). *If a set of modal formulae S is unsatisfiable, then there exists a closed tableau for S.*

3. Free-variable tableaux

In this section we introduce Tab_{FV}, i.e. free-variable tableau systems for the considered logics, and we show that they are sound and complete by showing that they are equivalent to the ground tableau calculi of Section 2. The definition of quantifier rules is similar to the classical case, but for the annotation on symbols that is taken into account in the definition of modal substitutions. First of all, we define a new extension of a language L, obtained by addition of *parameters* and *Skolem functions* to L^+:

DEFINITION 7. Let L be a modal language, $\mathcal{P} = \{z_1^{n_1}, z_2^{n_2}, ...\}$ an infinite denumerable set of new annotated symbols, called parameters, and $\mathcal{F}$ a denumerable set of new annotated function symbols such that, for each $i, j \in \mathbb{N}$, $\mathcal{P}$ contains infinitely many parameters of level i and $\mathcal{F}$ contains infinitely many function symbols of level i and arity j. Then the *labelled free-variable extension* of L is the language L^{fv} such that $L_P^{fv} = L_P$ and

$$L_F^{fv} = L_F^+ \cup \mathcal{P} \cup \mathcal{F}$$

The levels of parameters will be shown only when relevant. The definition of depth (Definition 3) extends to terms in L^{fv} with no modification. The next definition introduces the notion of modal substitution, where a parameter z can be mapped to a term t in L^{fv} only if the annotation of symbols in t "agrees" with the level of z. Since the agreement depends on which depths the term is at, the different DDE variants of QML correspond to different restrictions on modal substitutions.

DEFINITION 8. A modal substitution σ is a function from $\mathcal{P}$ to terms in L^{fv}, that is the identity almost everywhere, and such that if $\sigma(z^n) = t$ (and z has level n) then t is at depth n. Substitutions are denoted as usual by expressions of the form $\{t_1/z_1, \ldots, t_m/z_m\}$.

Unifiers, i.e. solutions of unification problems, and most general unifiers (*m.g.u.*) are defined like in the classical case. The following definition is also borrowed from classical tableau calculi.

DEFINITION 9. Let $n_1 : S_1, \ldots, n_k : S_k$ be all the leaves of a free-variable tableau $\mathcal{T}$, and, for each $i = 1, \ldots, k$, let P_i and $\neg Q_i$ be literals in $\mathcal{S}_i$. If the modal substitution σ is a solution of the unification problem $P_1 = Q_1, \ldots, P_k = Q_k$, then σ is an *atomic closure substitution for* $\mathcal{T}$. If it is a most general solution for such a unification problem, then it is a *most general atomic closure substitution for* $\mathcal{T}$.

Besides the propositional expansion rules, that are the same as in Tab_G (but for the fact that parameters and Skolem functions can occur in tableau nodes), the expansion rules of Tab_{FV} are:

Quantifier expansion rules:

$(\gamma)\ \dfrac{n : \forall x A, S}{n : A[z^n/x], \forall x A, S}$	$(\delta)\ \dfrac{n : \exists x A, S}{n : A[f^n(z_1^n, ..., z_k{}^n)/x], S}$

In the γ-rule, z is a new parameter. In the δ-rule, f^n is a new Skolem function, i.e. a symbol of level n in $\mathcal{F}$ that does not occur in $\{\exists x A\} \cup S$, and $z_1{}^n, ..., z_k{}^n$ are all the parameters with level n in $\exists x A, S$.

Most General Atomic Closure Substitution Rule: if $\mathcal{T}$ is a tableau for a set S of sentences in L and the modal substitution σ is a most general atomic closure substitution for $\mathcal{T}$, then the tableau $\mathcal{T}\sigma$, obtained by applying σ to $\mathcal{T}$, is a tableau for S.

Note that the Skolem term $f^n(z_1^n, ..., z_k{}^n)$ introduced by an application of the δ-rule contains only parameters with level n, and the Skolem function f is required to be "locally" new (i.e. new in the node). This is enough to ensure that the rule is sound. In fact, the role of the restriction on f and the parameters in the term is to prevent a modal substitution σ to unify $f^n(z_1^n, ..., z_k^n)$ and some other parameter z occurring in the premise of the δ-rule, $n : \exists x A, S$. If z has level n then $\sigma(f^n(z_1^n, ..., z_k{}^n)) = \sigma(z)$ is impossible, because z is one of $z_1^n, \ldots, z_k^n$. The level m of z cannot be greater than n, since a node never contains any symbol with level greater than its depth. Therefore, if $m \neq n$, then m is necessarily strictly less than n, and, by definition, a modal substitution cannot map a variable with level $m < n$ to a term containing a symbol of level n, that is not at depth m in any DDE variant of QML.

The definition of closed tableau (Definition 4) stays unchanged for Tab_{FV}. Note that in the calculi Tab_{FV}, closed tableaux are built up by first applying expansion rules, then computing an m.g.u. allowing to simultaneously close all the leaves, and applying the substitution rule just once. This makes the soundness proof simpler. However, like in the classical case, an alternative version of the rule can be considered, where branches can be closed one by one, by application of suitable most general unifiers. Since any most general atomic closure substitution can be seen as the composition of such "local" unifiers, this second version of the calculi is equivalent to the first one.

Here follow some sample Tab_{FV}-proofs. To start with, we give again a proof of the validity of **CBF** in the cumulative domain logics, using the π_K-rule, but this time in the calculus Tab_{FV}.

$$\cfrac{\cfrac{\cfrac{\cfrac{1 : \neg(\Box\forall x p(x) \to \forall x \Box p(x))}{1 : \Box\forall x p(x), \neg\forall x\Box p(x)}\,(\alpha)}{1 : \Box\forall x p(x), \neg\Box p(c^1)}\,(\delta)}{2 : \forall x p(x), \neg p(c^1)}\,(\pi_K)}{2 : p(z_1^2), \forall x p(x), \neg p(c^1)}\,(\gamma)$$

Here, c^1 is a (0-ary) Skolem function. Since c^1 is at depth 2, it suffices to apply the substitution $\{c^1/z_1^2\}$ to close the tableau. In the case of varying domain systems, such a substitution is illegal, because c^1 is not at depth 2.

As a further example, the tableau below is a Tab_{FV} proof of the validity of the formula $\Box\forall x\exists y p(x, y) \to \exists v\Box\exists w p(v, w)$ in the logic **K** with cumulative domains (since L_F is empty, there is no difference to be made about the designation of terms).

$$\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{1 : \neg(\Box\forall x\exists y p(x, y) \to \exists v\Box\exists w p(v, w))}{1 : \Box\forall x\exists y\; p(x, y),\; \neg\exists v\Box\exists w p(v, w)}\,(\alpha)}{1 : \Box\forall x\exists y\; p(x, y),\; \neg\Box\exists w p(z_1^1, w)}\,(\gamma)}{2 : \forall x\exists y\; p(x, y),\; \neg\exists w p(z_1^1, w)}\,(\pi_K)}{2 : \exists y\; p(z_2^2, y),\; \forall x\exists y\; p(x, y),\; \neg\exists w p(z_1^1, w)}\,(\gamma)}{2 : p(z_2^2, g^2(z_2^2)),\; \forall x\exists y\; p(x, y),\; \neg\exists w p(z_1^1, w)}\,(\delta)}{2 : p(z_2^2, g^2(z_2^2)),\; \forall x\exists y\; p(x, y),\; \neg\exists w p(z_1^1, w),\; \neg p(z_1^1, z_3^2)}\,(\gamma)$$

Since domains are cumulative, the term z_1^1 is at any depth $k \geq 1$, thus, in particular, it is at depth 2. Hence $\{z_1^1/z_2^2,\; g^2(z_1^1)/z_3^2\}$ is a modal substitution, and constitutes a most general atomic closure substitution. Its application yields a closed tableau. Note that in the varying domains option, no substitution can close the tableau, since z_1^1 is not at depth 2, and neither $\{z_1^1/z_2^2\}$ nor $\{z_2^2/z_1^1\}$ are legal substitutions. Indeed, the considered formula is not valid in this case.

3.1. Soundness of Tab_{FV}

Soundness and completeness of Tab_{FV} are established by showing the equivalence between the calculi Tab_{FV} and Tab_G, by means of syntactical transformation of Tab_G-tableau proofs into Tab_{FV}-tableau proofs and vice versa. Therefore, since Tab_G is sound and complete, a set S of modal formulae is unsatisfiable if and only if there is a closed Tab_{FV}-tableau for S.

In order to prove that any closed Tab_{FV}-tableau for S can be transformed into a closed Tab_G-tableau for S, a preliminary definition is needed.

Definition 10. Let L be a modal language and $E = \{t_1, .., t_n\}$ a set of terms in the language L^{fv}.

1. A *purification function* for E is an injective function ρ from the set of sub-terms of $t_1, ..., t_n$ to a set of constants from $\mathcal{C}$ not occurring in E, i.e. if $t_i \neq t_j$ then $\rho(t_i) \neq \rho(t_j)$.
2. Given a purification function ρ, the purification $P_\rho(E)$ of E is the set of terms obtained as follows:
   a) First, uniformly replace, in each term $t_i \in E$, all the occurrences of any *maximal* sub-term t of t_i, having a Skolem function f^n as its main symbol, by the constant $c = \rho(t)$ and annotate c with level n, thus obtaining a new set of terms E', possibly still containing parameters.
   b) Then uniformly replace any parameter z^n in E' by the dummy constant of level n.

   The element of $P_\rho(E)$ obtained from $t_i \in E$ will be denoted by $P_\rho(t_i)$ and called the "purification" of t_i.

For instance if $E = \{h^3(f^2(g^2(z^1)), z^1), g^2(z^1), f^2(z^1)\}$, h is a symbol of L, f, g are Skolem functions, and ρ maps $f^2(g^2(z^1))$ to c_1^2, $g^2(z^1)$ to c_2^2, $f^2(z^1)$ to c_3^2, then E' is $\{h^3(c_1^2, z^1), c_2^2, c_3^2\}$, and, in the case of varying domains and rigid designation when L contains at least a constant, $P_\rho(E) = \{h^3(c_1^2, a_0^0), c_2^2, c_3^2\}$; in the other cases $P(E) = \{h^3(c_1^2, a_1^1), c_2^2, c_3^2\}$.

Lemma 8. *If there exists a closed Tab_{FV} tableau for a set of modal sentences S, then there exists a closed Tab_G tableau for S.*

PROOF. Let $\mathcal{T}$ be a Tab_{FV}-tableau rooted at $n : S$ (with $n \geq 1$) and σ an atomic closure substitution for $\mathcal{T}$. If E is the set of terms occurring in $\mathcal{T}\sigma$ and ρ a purification function for E (mapping terms to new constants in $\mathcal{C}$), we prove that the tree $\mathcal{T}'$ obtained by uniformly replacing each term t in $\mathcal{T}\sigma$ with its purification $P_\rho(t)$ is a closed γ^*-tableau (γ^*-tableaux are defined in Section 2.2). The desired result then follows when $n = 1$, by application of Lemma 7, establishing that the γ^*-calculus is sound, knowing that Tab_G is sound. The proof is by induction on $\mathcal{T}$. Below, if $\mathcal{T}$ is any Tab_{FV}-tableau, $P_\rho(\mathcal{T})$ denotes the tableau obtained by replacing every term in $\mathcal{T}$ with its purification via ρ.

Base: if $\mathcal{T}$ is a leaf $n : S$ and $n : S\sigma$ contains a pair of complementary literals, then $P_\rho(n : S\sigma)$ is clearly a closed Tab_G-tableau, hence a γ^*-tableau.

Inductive Step: we consider different cases according to the first expansion rule applied to $\mathcal{T}$. We show here the treatment of some of them, the others being immediate.

1. The first expansion rule applied in $\mathcal{T}$ is the δ-rule, i.e. $\mathcal{T}$ has the form:

$$\frac{n : \exists x A, S}{n : A[f^n(z_1^n, ..., z_k{}^n)/x], S}$$
$$\vdots\, (\mathcal{T}_1)$$

By the inductive hypothesis, the tree $\mathcal{T}_1' = P_\rho(\mathcal{T}_1\sigma)$ is a closed γ^*-tableau. If $c^n = \rho(\sigma(f^n(z_1^n, ..., z_k{}^n))$ and $n : \exists x A', S' = P_\rho(n : \exists x A\sigma, S\sigma)$ is the purification of the node $n : \exists x A\sigma, S\sigma$, we show that c^n does not occur in $n : \exists x A', S'$. To start with, c^n cannot already appear in $n : \exists x A\sigma, S\sigma$, by construction. And c^n is not the purification of any term $\sigma(t)$ occurring in $n : \exists x A\sigma, S\sigma$. In fact, $c^n = P_\rho(\sigma(f^n(z_1^n, ..., z_k^n)))$, where $z_1, ..., z_k$ are all the parameters of level n occurring in $n : \exists x A, S$, and $\sigma(f^n(z_1, ..., z_k))$ does not occur in $n : \exists x A\sigma, S\sigma$, because:

- $f^n(z_1, ..., z_k)$ does not occur in $n : \exists x A, S$, since the Skolem function symbol f^n is new to $n : \exists x A, S$.
- there is no parameter z_i^m in $n : \exists x A, S$, such that $\sigma(z_i^m) = \sigma(f^n(z_1^n, ..., z_k^n))$. In fact, $m \neq n$, otherwise z_i^m would be among z_1^n, $\ldots$, z_k^n, so that $\sigma(z_i^m) = \sigma(f^n(z_1^n, ..., z_k^n))$ is impossible. Since a node with depth n cannot contain symbols labelled with levels greater than n, $m < n$. But this is again impossible, because, by definition of modal substitution, $\sigma(z^m)$ must be at depth m, and no matter which DDE-variant is considered, a term rooted at a function symbol of level $n > m$ cannot be at depth m.

Since, moreover, c^n has level n, the inference:

$$\frac{n : \exists x A', S'}{n : A'[c^n/x], S'}$$

is a correct application of the δ-rule in Tab_G, and the tree obtained by addition of such an inference on top of $\mathcal{T}_1'$ is the required closed γ^*-tableau $\mathcal{T}'$.

2. The first expansion rule applied in $\mathcal{T}$ is the γ-rule, and $\mathcal{T}$ is:

$$\begin{array}{c} \dfrac{n : \forall x A, S}{n : A[z^n/x], S} \\ \vdots\,(\mathcal{T}_1) \end{array}$$

   By the inductive hypothesis, the tree $\mathcal{T}_1' = P_\rho(\mathcal{T}_1\sigma)$ is a closed γ^*-tableau. The purification via ρ of

$$\frac{n : \forall x A\sigma, S\sigma}{n : A[z^n/x]\sigma, S\sigma}$$

   is clearly a correct application of the γ^*-rule. The result then follows from the inductive hypothesis.

3. $\mathcal{T}$ begins with an application of the β-rule, i.e. it has the form

$$\begin{array}{cc} \multicolumn{2}{c}{n : A \vee B, S} \\ \hline n : A, S & n : B, S \\ \vdots\,(\mathcal{T}_1) & \vdots\,(\mathcal{T}_2) \end{array}$$

   Clearly σ is also an atomic closure substitution for $\mathcal{T}_1$ and $\mathcal{T}_2$. If $\mathcal{T}_1'$ and $\mathcal{T}_2'$ are the closed γ^*-tableaux obtained by the inductive hypothesis, with respective roots $n : A', S'$ and $n : B', S'$, then the tableau

$$\begin{array}{cc} \multicolumn{2}{c}{n : A' \vee B', S'} \\ \hline n : A', S' & n : B', S' \\ \vdots\,(\mathcal{T}_1') & \vdots\,(\mathcal{T}_2') \end{array} \qquad \blacksquare$$

   is a closed γ^*-tableau.

The lemma above, together with Theorem 2, allows us to conclude:

THEOREM 4 (Soundness of Tab_{FV}). *If there exists a closed Tab_{FV} tableau for a set of modal formulae S, then S is unsatisfiable.*

3.2. Completeness of Tab_{FV}

Below, we show that the calculus Tab_{FV} is complete.

The syntactical transformation of ground tableaux into free-variable ones is carried out in the proof of the following lemma.

LEMMA 9. *If there is a closed Tab_G-tableau for a set of modal formulae S, then there is a closed Tab_{FV}-tableau for S.*

PROOF. We prove a more general result: let us assume that $\mathcal{T}_1$ is a tree, labelled by sets of formulae in L^{fv}, for some modal language L, and that θ is a modal substitution such that $\mathcal{T}_1\theta$ is a closed Tab_G-tableau. Then there exists a Tab_{FV}-tableau $\mathcal{T}_2$, with the same root as $\mathcal{T}_1$, and an atomic closure substitution σ for $\mathcal{T}_2$. The desired result corresponds to the particular case where $\mathcal{T}_1$ contains no parameters and Skolem functions (i.e. its nodes contain formulae in L^+), $\theta = \emptyset$ and $n = 1$.

The definition of $\mathcal{T}_2 = Tr(\mathcal{T}_1)$ and σ is inductive. In the base case, $\mathcal{T}_1$ is a leaf $n : S$, such that $n : S\theta$ is closed. Then we take $Tr(\mathcal{T}_1) = \mathcal{T}_1$ and $\sigma = \theta$.

In the inductive steps, different cases are considered, according to the first expansion rule applied in $\mathcal{T}_1\theta$. We show here the treatment of some of them.

1. Let $\mathcal{T}_1\theta$ start with an application of the γ-rule, i.e. $\mathcal{T}_1$ has the form:

$$\frac{n : \forall x A, S}{n : A[t/x], \forall x A, S}$$
$$\vdots\ (\mathcal{T})$$

where $t \in \mathrm{dom}(n : \forall x A\theta, S\theta)$. Let $\mathcal{T}'$ be the tree obtained from $\mathcal{T}$ by replacing every occurrence of t, descending from the displayed occurrence of t in $A[t/x]$, with a new parameter z^n, annotated with level n. Such a tree is rooted at

$$n : A[z^n/x], \forall x A, S.$$

If $\theta' = \theta \cup \{t/z^n\}$, then clearly θ' is a modal substitution such that $\mathcal{T}'\theta'$ is a closed Tab_G-tableau. Therefore, the induction hypothesis can be applied, giving a free-variable tableau $Tr(\mathcal{T}')$, rooted at $n : A[z^n/x], \forall x A, S$ and an atomic closure substitution σ for $Tr(\mathcal{T}')$. Then $Tr(\mathcal{T}_1)$ is

$$\frac{n : \forall x A, S}{n : A[z^n/x], \forall x A, S}$$
$$\vdots\ (Tr(\mathcal{T}'))$$

and σ is the required substitution.

2. Let $\mathcal{T}_1\theta$ start with an application of the δ-rule, i.e. $\mathcal{T}_1$ has the form:

$$\frac{n : \exists x A, S}{n : A[c^n/x], S}$$
$$\vdots\ (\mathcal{T})$$

where c^n is new to $\exists x A\theta, S\theta$. Let us consider the tree $\mathcal{T}'$, obtained from $\mathcal{T}$ by replacing each occurrence of c^n by $f^n(z_1, ..., z_k)$, where $z_1, ..., z_k$ are all the parameters of level n in $\exists x A, S$. Such a tree is rooted at

$$n : A[f^n(z_1, ..., z_k)/x], S.$$

Clearly, $\mathcal{T}'\theta$ is a closed Tab_G-tableau, so that the induction hypothesis can be applied, giving a free-variable tableau $Tr(\mathcal{T}')$ rooted at $n : A[f^n(z_1, ..., z_k)/x], S$, and an atomic closure substitution σ for $Tr(\mathcal{T}')$. Then $Tr(\mathcal{T}_1)$ is

$$\frac{n : \exists x A, S}{n : A[f^n(z_1, ..., z_k)/x], S}$$
$$\vdots\ (Tr(\mathcal{T}'))$$

and its atomic closure substitution is σ.

Therefore, if there is a closed Tab_G-tableau for S, there is also a Tab_{FV}-tableau $\mathcal{T}$ for S and an atomic closure substitution σ for $\mathcal{T}$. If σ is not the most general one, then we conclude by observing that if there is an atomic closure substitution for $\mathcal{T}$, then there is also a most general one. ∎

From the lemma above and Theorem 3, it follows:

THEOREM 5 (Completeness of Tab_{FV}). *If a set of modal formulae S is unsatisfiable, then there exists a closed Tab_{FV}-tableau for S.*

4. Concluding remarks

In this paper we provide calculi treating different variants of first-order modal logic (varying and cumulative domains, rigid and non-rigid designation, local and non-local terms), by addition of a minimal semantical annotation to tableau methods obtained by extending propositional modal systems with the classical rules for quantifiers. We give both a ground formulation of the calculi (Tab_G) and free variable tableau systems (Tab_{FV}). In both cases, the different DDE variants are obtained in a simple and modular way, just by choosing among alternative definitions of a few basic operations. In particular, in the case of Tab_{FV}-calculi, a first-order variant is essentially characterized by the notion of modal substitution. Differently from other approaches such as for instance [17, 2], however, term unification does not depend on the propositional kernel of the logic (i.e. on the accessibility relation) and does not involve any complex or strongly semantically dependent operation. The only semantically motivated operations that are additionally needed, with respect to classical unification, are simple numerical "greater than" tests.

Although we have treated each DDE variant as a different logic, some mixed approaches are possible with respect to the denotation of symbols: for example, it can be established that constant symbols (proper names) are rigid while functional symbols are not, so that definite descriptions do not necessarily denote rigidly. Each symbol is then to be treated according to the corresponding variant in the construction of the initial tableau and in the application of dynamic rules.

In the case of the (proof-theoretically) easiest DDE variant assuming cumulative domains, rigid designation and local terms, the calculi Tab_G are actually the same as what is obtained by a simple addition of classical quantifier rules to modal propositional ones (such as, for example, in [11], Chapter 7). However, their Tab_{FV} counterpart is not exactly the same as proposed in [12], where a tableau calculus with free variables, dynamic skolemization and unification is defined, that is correct and complete for this variant of QML. There, the new Skolem term $f(z_1, ..., z_n)$ introduced by an application of the δ-rule is such that $z_1, \ldots, z_n$ are all the parameters occurring in the branch. This rule guarantees that if $A(z)$ and $A(t)$ are unifiable, and the term t has been introduced by an application of the δ-rule, then such an application precedes the application of the γ-rule that introduces z in the branch. So, in particular, it cannot happen that a γ-rule introduces the parameter z varying over the domain of a given world w, and t refers to an object existing in a world accessible from w, but not necessarily in w. Exploiting the annotation on symbols, the δ-rule in Tab_{FV} is somewhat more liberal, requiring only parameters having the same level as the depth of the node to be the arguments of the Skolem functions. Probably, the δ-rule can be further relaxed, in the style of [16, 6, 4]. The details, however, still have to be worked out.

The (ground) prefixed tableaux presented in Chapter 8 of [11], in the three different variants of constant, cumulative and varying domains, associate prefixes with constants introduced by applications of the δ-rule in a way that bears some resemblance with Tab_G and its labelling mechanism. The present approach, however, is not in the style of prefixed tableaux, where prefixes are associated to each formula (not the whole set of "active" formulae), are more complex structures than integers and play a fundamental role in modal expansion rules. A stricter relation links the present work with [8, 9], where the same kind of annotation is used to restrict unification in the context of modal resolution. However, since the Skolem-like transformation is static in that context, it works well with logics such as **K** and **D**, where formulae cannot move from their modal context, but requires a preliminary propositional transformation of formulae when dealing with reflexive and/or transitive logics, where the modal degree of a formula is subject to change. The dynamic introduction of parameters and Skolem terms in tableau systems makes things easier: a quantifier is treated only when it comes to the surface, i.e. when, after possible changes due to applications of the ν_T or π_4-rule, its modal context is settled.

References

[1] Abadi, M., and Z. Manna: 1986, 'Modal Theorem Proving', In: *Proc. of the 8th Int. Conf. on Automated Deduction*, Berlin, pp. 172–189.

[2] Artosi, A., P. Benassi, G. Governatori, and A. Rotolo: 1998, 'Shakespearian modal logic: A labelled treatment of modal identity', in: M. Kracht, M. de Rijke, H. Wansing, and M. Zakharyaschev (eds.), *Advances in Modal Logic*, Stanford: CSLI Publications, pp. 1–21.

[3] Auffray, Y., and P. Enjalbert: 1992, 'Modal Theorem Proving: an equational viewpoint', *Journal of Logic and Computation* 2, 247–297.

[4] Baaz, M., and C. Fermüller: 1995, 'Non-elementary speedups between different versions of tableaux', in: *Proc. of Tableaux 95*, pp. 217–230.

[5] Basin, D., M. Matthews, and L. Viganò: 1998, 'Labelled Modal Logics: Quantifiers', *Journal of Logic Language and information* 7(3), 237–263.

[6] Beckert, B., R. Hähnle, and P.H. Schmitt: 1993, 'The even more liberalized δ-rule in free variable semantic tableaux', in: *Proc. of the 3rd Kurt Gödel Colloquium KGC'93*, pp. 108–119.

[7] Belnap, N.: 1982, 'Display Logic', *J. of Philosophical Logic* 11, 375–417.

[8] Cialdea, M.: 1991, 'Resolution for some first order modal systems', *Theoretical Computer Science* 85, 213–229.

[9] Cialdea, M. and L. Fariñas del Cerro: 1986, 'A Modal Herbrand's Property', *Z. Math. Logik Grundlag. Math.* 32, 523–539.

[10] Cialdea Mayer, M., and S. Cerrito: 2000, 'Variants of First-Order Modal Logics', in: R. Dyckhoff (ed.), *Proc. of Tableaux 2000.*

[11] FITTING, M.: 1983, *Proof Methods for Modal and Intuitionistic Logics*, Reidel Publishing Company.

[12] FITTING, M.: 1988, 'First-Order Modal Tableaux', *Journal of Automated Reasoning* 4, 191–213.

[13] FITTING, M.: 1999, 'On quantified modal logic', *Fundamenta informaticæ* 39, 105–121.

[14] GARSON, J. W.: 1984, 'Quantification in modal logic', in: D. Gabbay and F. Guenthner (eds.): *Handbook of Philosophical Logic*, Vol. II, D. Reidel Publ. Co., pp. 249–307.

[15] GORÉ, R.: 1999, 'Tableau methods for modal and temporal logics', in: M. D'Agostino, G. Gabbay, R. Hähnle, and J. Posegga (eds.): *Handbook of tableau methods*, Kluwer.

[16] HÄHNLE, R. and P. H. SCHMITT: 1994, 'The liberalized δ-rule in Free variable semantic tableaux', *Journal of Automated Reasoning* 13, 211–222.

[17] JACKSON, P. and H. REICHGELT: 1987, 'A general proof method for first-order modal logic', in: *Proc. of the 10th Joint Conf. on Artificial intelligence (IJCAI '87)*, pp. 942–944.

[18] KONOLIGE, K.: 1986, 'Resolution and quantified epistemic logics', in: J. H. Siekmann (ed.): *Proc. of the 8th int. Conf. on Automated Deduction (CADE 86)*, pp. 199–208.

[19] MINTS, G.: 1997, 'Indexed systems of sequents and cut-elimination', *J. of Philosophical Logic* 26, 671–696.

[20] OHLBACH, H. J.: 1991, 'Semantics Based Translation Methods for Modal Logics', *Journal of Logic and Computation* 1(5), 691–746.

[21] WALLEN, L. A.: 1990, *Automated Deduction in Nonclassical Logics: Efficient Matrix Proof Methods for Modal and intuitionistic Logics*, MIT Press.

[22] WANSING, H.: 1962, 'Predicate logics on display', *Studia Logica* 62(1), 49–75.

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