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On Orthogonal 3D Shapes of Theta Graphs

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RT-DIA-71-2002 **Giugno 2002**

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This work was partially supported by European Commission - Fet Open project COSIN IST-2001-33555 and by "Progetto Giovani Ricercatori: Algoritmi per la Visualizzazione di Grafi di Grandi Dimensioni" of the University of Perugia, Italy.

ABSTRACT

The recent interest in three dimensional graph drawing has been motivating studies on how to extend two dimensional techniques to higher dimensions. A common 2D approach for computing an orthogonal drawing clearly separates the task of defining the shape of the drawing from the task of computing its coordinates. First results towards finding a three-dimensional counterpart of this approach are presented in [11, 12], where characterizations of orthogonal representations of paths and cycles are studied. In this note we show that the known characterization for cycles does not immediately extend to even seemingly simple graphs such as theta graphs. A sufficient condition for recognizing three-dimensional orthogonal representations of theta graphs is also presented.

1 Introduction

The recent interest in three-dimensional graph drawing has been motivating studies on how to extend two dimensional techniques to 3D space. Work in this direction includes extensions of simulated annealing techniques, spring embedder techniques, and incremental techniques (see e.g., [5, 9, 15, 22, 23, 28]). However, while a rich body of literature is devoted to three-dimensional orthogonal drawings (see e.g. [3, 8, 14, 16, 17, 21, 23, 29]), little is known on the challenging task of extending to 3D the well-known topology-shape-metrics approach [26].

The topology-shape-metrics approach for two-dimensional space consists of three main steps: In the first step a planar embedding of the input graph G is defined. In the second step a two-dimensional orthogonal representation of G is computed. An orthogonal representation is an equivalence class of orthogonal drawings of G all having the same shape and such that no two edges intersect. It can be described by labeling each edge (u, v) of G with a sequence of labels in the set $\{East, West, North, South\}$. In the third step, the coordinates for the nodes and for the bends along the edges are found.

A key component of the two-dimensional topology-shape-metrics technique is a characterization of orthogonal representations, that is the properties that must be satisfied by the labeling in order to guarantee the existence of an orthogonal drawing where no two edges intersect. Such a characterization can be found in the works by Vijaian and Widgerson and by Tamassia [26, 27].

Extending to three-dimensional space the topology-shape-metrics approach implies computing the shape and the coordinates of the drawing in two different steps. Also, a three-dimensional counterpart of the characterization by Vijaian and Widgerson and by Tamassia is needed. More precisely, a solution to the following problem has to be found: Let G be a graph whose edges are directed and labeled with a sequence of labels in the set $\{Up, Down, East, West, North, South\}$; does a three-dimensional orthogonal drawing of G exist such that each edge has a shape consistent with its labeling and no two edges intersect?

This question has been addressed by Di Battista et al. [11, 13, 12] for simple classes of graphs, namely paths and cycles. Let π be a path whose edges have labels in the set $\{Up, Down, East,\ West,\ North, South, \}$ and let p and q be two points in 3D space. In [11, 13] it is characterized when π admits an orthogonal three-dimensional drawing Γ such that: (i) Γ starts at p and ends at q, (ii) the edges of Γ follow the directions given by the labeling, and (iii) no two edges of Γ intersect. The result is then extended in [12], where a characterization of three-dimensional orthogonal representations of cycles is given.

The goal of this note is to shed some more light on the above basic question. Our results can be listed as follows.

- We show that the known characterization for cycles does not immediately extend to even seemingly simple graphs such as theta graphs.
- We give a sufficient condition for recognizing three-dimensional orthogonal representations of theta graphs.
- We present an algorithm that computes a three-dimensional orthogonal drawing from an orthogonal representation of a theta graph that satisfies the above condition.

We remark that theta graphs have been studied extensively in the literature. For example, they arise in problems concerning graph planarity (see, e.g., [6, 2, 25, 1]), graph bandwidth (see, e.g., [20, 24, 7, 18]), and chromatic polynomials (see, e.g., [4]).

The remainder of this paper is organized as follows. In Section 2 some preliminary definitions are given. Section 3 shows that the characterization for cycles does not immediately extend to even seemingly simple graphs such as theta graphs. In Section 4 we introduce some result that are used in Section 5 to prove the sufficiency of the condition for the existence of a three-dimensional orthogonal drawing from an orthogonal representation of a theta graph.

2 Preliminaries

We assume familiarity with basic graph drawing terminology (see, e.g. [10]).

A *direction label* is a label in the set $\{U, D, E, W, N, S\}$ specifying the directions *Up*, *Down*, *East*, *West*, *North*, *South*, respectively.

Given a graph G, for each (undirected) edge e of G with endpoints u and v, we call *darts* the two possible orientations (u, v) and (v, u) of edge e. A *3D shape graph* γ is a labeling of the darts of G such that (i) each dart is associated with a direction label; (ii) the two darts of the same edge have opposite labels; (iii) γ contains at least one of each oppositely directed pair of directions (*truly-threedimensionality*); and (iv) each node does not have two entering darts with the same label (*coherence*).

In case of 3D shape paths and cycles, which have been studied in [11, 13] and [12], respectively, an arbitrary uniform orientation for the edges can be chosen in order to describe the 3D shape as a sequence (a circular sequence, for cycles) of labels.

A *theta graph* is a graph with two non adjacent nodes of degree three and all other nodes of degree two [19]. Thus, a theta graph consists of two nodes of degree three and three disjoint paths, of length at least two, joining them.

We will call *3D theta shape* a 3D shape for a theta graph. For example, Figure 1 shows a theta graph Θ and two different labelings of the darts of Θ. The labeling of Figure 1.b is a theta shape, while the labeling of Figure 1.c is not a theta shape, since two consecutive labels have opposite direction.

In the following we will denote by p and q the two degree-three nodes of a theta shape. Also, given a shape path π_x from p to q, and a shape path π_y from q to p, we denote by $C_{x,y}$ the shape cycle obtained by joining π_x and π_y. Observe that given a theta shape, three paths π_x, π_y, and π_z, and three cycles $C_{x,y}$, $C_{x,z}$, and $C_{y,z}$, are defined.

A *three dimensional orthogonal drawing* of a graph is such that nodes are mapped to grid points of an integer three dimensional grid and edges are segments along the integer grid lines connecting the end points.

An *intersection* in a three dimensional orthogonal drawing is a pair of edges that overlap in at least one point that does not correspond to a common end-node. Figure 2 shows some example.

The *simplicity testing* problem for a 3D shape graph γ is to decide whether there exists an orthogonal drawing Γ of γ such that no two edges of Γ intersect and each oriented edge satisfies the direction constraint defined by the direction labels associated with its darts. Obviously, a graph of degree greater than six is such that all 3D shape graphs defined on it are not simple.

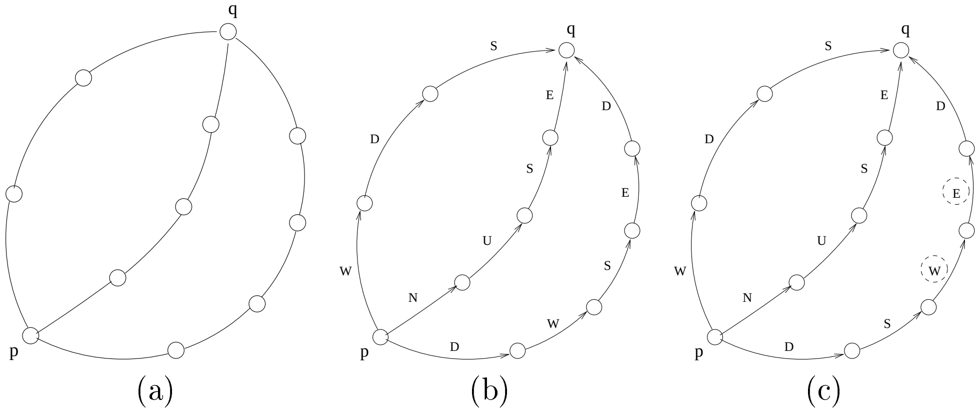

Figure 1: (a) A theta graph Θ. (b) A labeling of Θ that is a theta shape. (c) A labeling of Θ that is not a theta shape because two consecutive labels (the circled ones) have opposite direction.

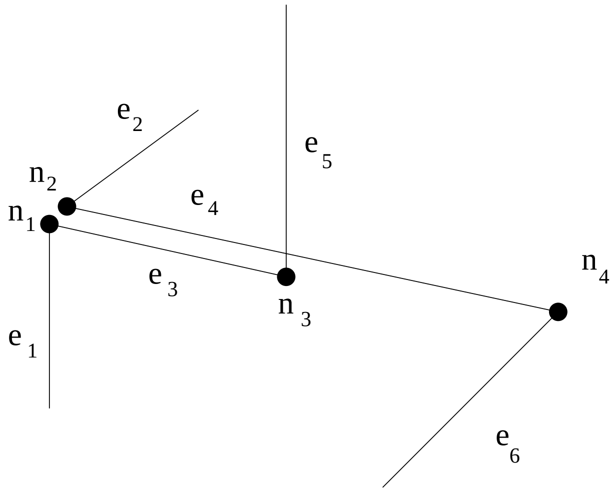

Figure 2: An orthogonal drawing showing five intersections: (e_1, e_2), (e_1, e_4), (e_2, e_3), (e_3, e_4), and (e_4, e_5). Overlapping objects are drawn very near (for example nodes n_1 and n_2 or edges e_3 and e_4)

Let γ be a 3D shape graph. A *flat* of γ is 3D shape subgraph of γ that is maximal with respect to the property that its labels come from the union of two oppositely directed pairs of directions. Observe that, any orthogonal drawing of a flat F must consist of edges that lie on the same axis-aligned plane. Also, the definition above extends the analogous definitions given in [11, 13, 12].

The following lemmas will be useful in the remainder of the paper.

Lemma 1 *A shape graph γ that admits a three dimensional orthogonal drawing such that no intersection occurs between two edges of the same flat is simple.*

Proof: We prove the statement by construction. Namely, given a three dimensional orthogonal drawing Γ such that no intersection occurs between two edges of the same flat, we define an iterative process that produces an orthogonal drawing Γ' without intersections. Consider an intersection between two edges e_1 and e_2 that do not belong to the same flat. We describe a technique that eliminates the intersection, without introducing a new one. Consider a plane Π common to e_1 and e_2 and a direction d orthogonal to Π. Move one unit in the d direction all the nodes in the open half-space determined by Π and d, the end-points of e_1, and the end-points of all the edges of the flat F of e_1 perpendicular to d, if any.

The obtained drawing is a 3D orthogonal drawing. In fact, the end-points of each edge orthogonal to d may have been either both moved one unit in the d direction or both left in their original position. Thus, in the new drawing edges orthogonal to d are axis aligned and of the same length they had in the original drawing. None, one, or both the end-points of each edge parallel to d may have been moved in the d direction. If none or both end-points have been moved, then the edge has its original direction and length. If only one end-point has been moved, then the other end-point must be in the closed half-space determined by Π and $\overline{d}$, where $\overline{d}$ denotes the direction opposite to d, and thus the edge is longer than it was in the original drawing and has the same axis-aligned direction.

The intersection has been removed. In fact, e_1 lays on a new plane Π' parallel to Π, while edge e_2 has not been moved from Π, since by hypothesis it does not belong to F. Now we show that no other intersection has been introduced. Observe that the open half-space determined by Π and $\overline{d}$ has not been changed. Also, no intersection is introduced in the open half-space determined by Π' and d, which has been shifted towards d. The only region that has been changed is the closed region between Π and Π'. In particular, an intersection may lay either on Π or on Π'. Consider an intersection between two edges laying both on Π. Since such edges have not been moved, such intersection was already present in the drawing. Consider an intersection between an edge laying on Π and an edge orthogonal to Π. The latter edge has an end-point on the closed half space determined by Π and $\overline{d}$. Since such end-point has not been moved the intersection was already present in the drawing. Edges on Π' belong to the same flat F of e_1 and by hypothesis they can not cross each other. Consider an intersection between an edge laying on Π' and an edge orthogonal to Π'. The latter edge has an end-point on the closed half space determined by Π and $\overline{d}$ or on Π'. In both cases, the intersection was already present when the edge was on Π. □

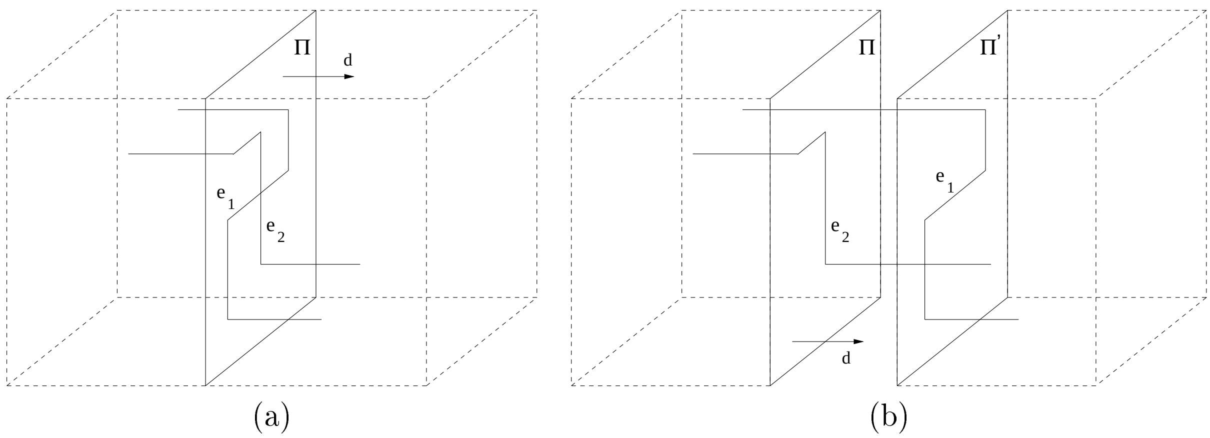

Figure 3: The removal of an intersection: (a) before the removal (b) after the splitting operation.

By virtue of the preceding lemma, in the following sections we will neglect intersections occurring between edges (and nodes) that do not share a flat, assuming that they could be easily eliminated in a post processing step.

Lemma 2 *Let γ be a shape graph and v be a node with two incident edges e_1 and e_2, such that e_1 enters v and e_2 leaves v with the same label X. The shape graph γ is simple if and only if the shape graph γ', obtained from γ by replacing the path composed by e_1 and e_2 with a single edge e directed as e_1 and e_2 and with direction label X, is simple.*

Proof: Suppose γ is simple. Consider a simple 3D orthogonal drawing of γ, replace the two segments corresponding to edges e_1 and e_2 with a single segment. The obtained drawing is a simple 3D orthogonal drawing of γ'. Suppose γ' is simple. Consider a simple 3D orthogonal drawing of γ'. If the segment corresponding to edge e has length $l > 1$, replace it with two segments of length 1 and $l - 1$, respectively. Otherwise, multiply all coordinates by two and apply the previous strategy. The obtained drawing is a simple 3D orthogonal drawing of γ. □

By virtue of Lemma 2, in the following we will restrict our attention to shape graphs that do not have nodes of degree two whose incident edges are parallel to the same axis. A similar hypothesis has been made in [11, 13] and [12] for the case of paths and cycles described by sequences of direction labels (that is, consecutive labels do not define the same direction).

Lemma 3 *Let γ be a simple shape graph. Let e_1 and e_2 be two edges leaving node v with orthogonal direction labels X_1 and X_2, respectively. If the flat of e_1 and e_2 is acyclic and no edge leaving v has direction label Y, with Y orthogonal to X_1 and X_2, then the shape graph γ', obtained from γ by replacing e_1 with a path of two edges e_1' and e_1'' directed away from v and with labels Y and X_1, respectively, is simple.*

Proof: Suppose γ is simple. Consider a simple 3D orthogonal drawing of γ. Call Π the plane common to e_1 and e_2, F the flat of e_1 perpendicular to Y, v_1 the node opposite to v with respect of e_1, and l_1 the length of e_1. Denote by V^* the nodes of F joined with v_1 with a path in F not containg e_1. Since by hypothesis F is acyclic $v \notin V^*$. Remove the segment from v to v_1 corresponding to e_1. Move one unit in the Y direction all the nodes in V^* and all the nodes in the open half-space determined by Π and Y. Connect v to v_1 with two segments of length 1 and l_1 and direction Y and X_1, respectively.

It could be easily shown that the obtained drawing is a simple 3D orthogonal drawing of γ'. In fact, the two segments added to the drawing have direction coherent with the labels of e_1' and e_1''. □

Given a 3D shape path or cycle described by a sequence of direction labels σ such that no adjacent labels are equal, a not necessarily consecutive subsequence $\tau \subseteq \sigma$, where τ consists of k elements, is a *canonical sequence* provided that:

- $1 \leq k \leq 6$;
- the labels of τ are distinct;
- no flat of σ contains more than three labels of τ; and
- if a flat F of σ contains one or more labels of τ, then $\tau \cap F$ form a consecutive subsequence of σ.

A characterization of simple shape cycles is given in [12]:

Theorem 1 [12] *A 3D shape cycle described by a sequence of direction labels σ such that no adjacent labels are equal is simple if and only if it contains a canonical sequence of length six.*

Figures 4 and 5 show a simple shape cycle and a not simple one, respectively.

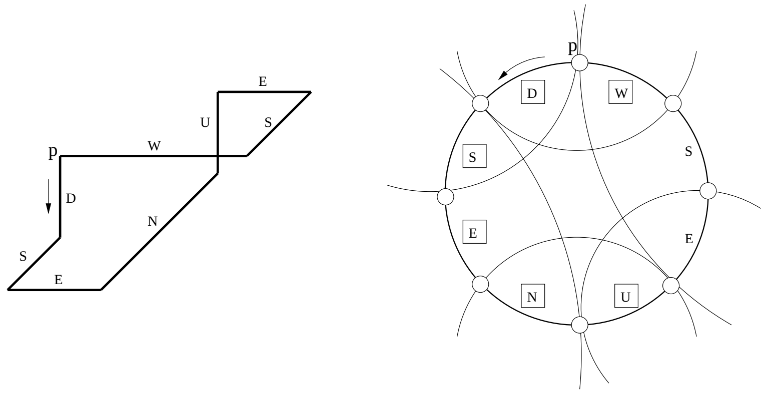

Figure 4: A simple shape cycle meeting the conditions of Theorem 1.

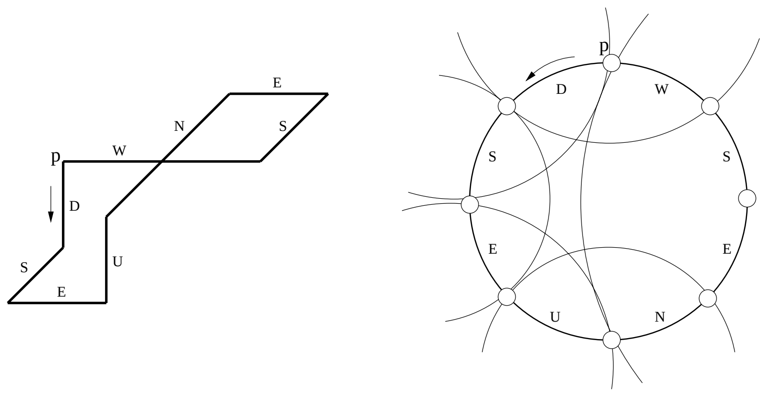

Figure 5: A not simple shape cycle that does not meet the conditions of Theorem 1.

3 A Forbidden Theta Shape

In this section we show that the simplicity of the cycles composing a theta shape does not imply the simplicity of the theta shape itself. We use the following notation: given a shape path π (say $DWUNE$) we will denote by $\overline{\pi}$ ($\overline{DWUNE}$ in the example) the shape path obtained by orienting each edge in the opposite direction and changing its label with the opposite one ($\overline{DWUNE} = WSDEU$). Also, we will use a dot to denote a series composition of paths (for example $\pi_1 = \pi_2 \cdot \overline{\pi_3}$). Finally, when we will need to identify the nodes between two edges of the sequence, we will insert a lower letter between the labels corresponding to the edges (as in $pWbNUaEDq$). The same notations apply for cycles, which are circularly ordered sequences of direction labels.

Given two distinct nodes v and w of a shape path (shape cycle, theta shape, respectively) γ, we say that v *is* Y *with respect to* w, where $Y \in \{U, D, E, W, N, S\}$, if in any drawing of γ, denoted Π_v and Π_w the two planes orthogonal to Y containing v and w respectively, the two planes may be joined by a segment oriented from Π_w to Π_v which has direction Y. Observe that, if v is Y with respect to w, w is $\overline{Y}$ with respect to v.

The following lemmas hold:

Lemma 4 *Given a shape path σ connecting two points p and q, if σ contains only one (say X) of an oppositely directed pair X, Y of direction labels, then in any drawing of σ p is Y with respect to q, or, equivalently, q is X with respect to p.*

Lemma 5 *Given a shape path σ connecting two points p and q, if σ does not contain labels of an oppositely directed pair of direction labels, then in any drawing of σ its edges lay on the same plane.*

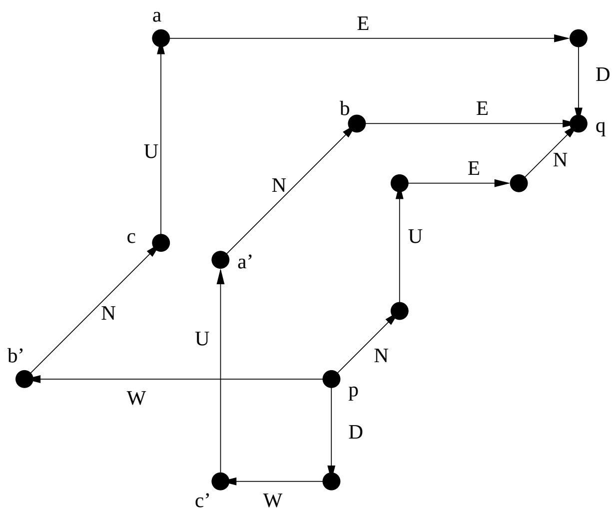

Figure 6: A forbidden theta shape.

Theorem 2 *There exists a 3D shape graph that is not simple even if all its induced cycles are simple.*

Proof:

Let Θ be the theta shape composed by the three paths: $\pi_1 = WNUED$, $\pi_2 = DWUNE$, and $\pi_3 = NUEN$ (see Fig. 6). The three cycles $C_{1,2}$, $C_{1,3}$, and $C_{2,3}$ defined by Θ are simple. We have:

$$C_{1,2} = WNUED \cdot \overline{DWUNE} = WNUEDWSDEU$$

$$C_{1,3} = WNUED \cdot \overline{NUEN} = WNUEDSWDS$$

$$C_{2,3} = DWUNE \cdot \overline{NUEN} = DWUNESWDS$$

Each of them satisfies the hypotheses of Theorem 1. In fact, $C_{1,2}$ contains, for example, the canonical sequence identified by the checked labels in the sequence $\check{W}\check{N}UE\check{D}W\check{S}D\check{E}\check{U}$; $C_{1,3}$ contains, for example, the canonical sequence identified by the checked labels in the sequence $W\check{N}\check{U}\check{E}D\check{S}\check{W}\check{D}S$; and $C_{2,3}$ contains, for example, the canonical sequence identified by the checked labels in the sequence $\check{D}\check{W}\check{U}\check{N}\check{E}\check{S}WDS$.

Suppose for a contradiction that Θ is simple. Consider any orthogonal drawing Γ of Θ. In order to identify some nodes we will describe the three cycles as

$$C_{1,2} = pWb'NcUaEDqWbSa'Dc'EU$$

$$C_{1,3} = pWb'NcUaEDqSWDS$$

$$C_{2,3} = pDWc'Ua'NbEqSWDS$$

Observe that:

1. from $a'NbEqSWDSp$ and from Lemma 4 it follows that in Γ a' is U with respect to p.
2. from $pDWc'$ and from Lemma 4 it follows that c' is DW with respect to p.
3. consider the two orthogonal edges $e_1 = a'Dc'$ and $e_2 = pWb'$. From $a'Dc'EUpWb'$ and from Lemma 5 it follows that e_1 and e_2 belong to the same flat.
4. from statements 1, 2, and 3 and from the hypothesis that Θ is simple, we have that b' must be E with respect to a' and c'.
5. from $aEDq$ and from Lemma 4 it follows that a is UW with respect to q
6. from $bEqSWDSpWb'Nc$ and from Lemma 4 it follows that c is D with respect to b
7. consider the two orthogonal edges $e'_1 = aDc$ and $e'_2 = qWb$. From $cUaEDqWb$ and from Lemma 5 it follows that e'_1 and e'_2 belong to the same flat.
8. from statements 5, 6 and 7, and from hypothesis that Θ is simple we have that b must be E with respect to a and c in order to avoid an intersection.
9. from $bSa'Dc'$ follows that b and c' share the same plane orthogonal to the EW direction.
10. from $b'NcUa$ follows that b' and a share the same plane orthogonal to the EW direction.

By using 9, 10 we obtain from statement 8 that c' must be E with respect to b', which is in contradiction with statement 4. $\square$

4 Triply Expanding Drawings

Let $\pi_1, \pi_2, \ldots, \pi_n$ be n shape paths starting from a common point p. Denote by m_i the number of edges of π_i, and by $e_{i,h}$, with $h = 1, \ldots, m_i$, the h-th edge of π_i starting from p.

An *expanding drawing* of $\pi_1, \pi_2, \ldots, \pi_n$ is a simple 3D orthogonal drawing for which edges $e_{1,m_1}, e_{2,m_2}, \ldots, e_{n,m_n}$ can be replaced by arbitrarily long segments without creating any intersection with the drawing. The *bounding box* of an expanding drawing is the bounding box of the drawing when edges $e_{1,m_1}, e_{2,m_2}, \ldots, e_{n,m_n}$ are removed.

In [11, 13] Di Battista et al. showed a sufficient condition for the existence of an expanding drawing of two paths (*doubly expanding drawing*). In this section we will show a sufficient condition for the existence of an expanding drawing of three paths (*triply expanding drawing*).

The result of [11, 13] on doubly expanding drawing is the following:

Theorem 3 [11, 13] *A shape path π with n edges admits a doubly expanding drawing if either it consists of exactly two edges or it contains at least two flats.*

A doubly expanding drawing may be built, for example, with the technique described in the proof of Theorem 3 given in [11, 13] that here we briefly recall. Denote e_i, with $i = 1, \ldots, n$, the edges of π. If $n = 2$ then the construction is trivial. Otherwise, since π contains more than one flat, there must exists an edge common to two flats. Let k be the minimum index for which e_k is common to two flats. Draw e_k as a segment of length one with the tip at the origin. Add e_j, with $j = k-1, \ldots, 1$ in such a way that, when a new edge is drawn, it extends farther by one in the direction opposite to the one associated with it than any previously drawn segment. Add e_j, with $j = k+1, \ldots, n$ in such a way that, when a new edge is drawn, it extends farther by one in its direction than any previously drawn segment.

Consider three paths π_x, π_y, and π_z starting from a common point p. If $e_{i,1}$ and $e_{j,1}$ are orthogonal, denote by $F_{i,j}$ the flat of $\overline{\pi_i} \cdot \pi_j$ which contains the labels associated with $e_{i,1}$ and $e_{j,1}$. If $e_{x,1}$, $e_{y,1}$, and $e_{z,1}$ are pairwise orthogonal, p belongs to the three orthogonal flats $F_{x,y}$, $F_{y,z}$, and $F_{x,z}$. If the number of edges of path π_i on flat $F_{i,j}$ is greater than one, we say that π_i *has its tail on* $F_{i,j}$. If $e_{x,1}$, $e_{y,1}$, and $e_{z,1}$ are pairwise orthogonal and $m_i > 1$ then π_i has its tail on either $F_{i,j}$ or $F_{i,k}$. A flat $F_{i,j}$ may host the tail of π_i and π_j. See Figure 7 for an example.

Lemma 6 *Let π_x, π_y, and π_z be three shape paths starting from a common point p, such that: (i) $e_{x,1}$, $e_{y,1}$, and $e_{z,1}$ are pairwise orthogonal and (ii) the paths π_x, π_y, and π_z contain each a single flat. If the paths $\overline{\pi_x} \cdot \pi_y$, $\overline{\pi_x} \cdot \pi_z$, and $\overline{\pi_y} \cdot \pi_z$ consist of exactly two edges or contain at least two flats, then π_x, π_y, and π_z admit a triply expanding drawing.*

Proof: Observe that if $m_i = 1$ for a path π_i, then the other two paths π_j and π_k can not have their tails on flats $F_{i,j}$ and $F_{i,k}$ otherwise $\overline{\pi_i} \cdot \pi_j$ or $\overline{\pi_i} \cdot \pi_k$ would consist of more than two edges and would contain one flat only, contradicting the hypotheses. Further, if $m_i > 1$ and $m_j > 1$ for two paths π_i and π_j, then their two tails can not lay both on the flat $F_{i,j}$, otherwise $\overline{\pi_i} \cdot \pi_j$ would consist of more than two edges and would contain one flat only, contradicting the hypotheses. Thus, either $m_i > 1$, for all $i = x, y, z$, or $m_i = 1$, for all $i = x, y, z$. In fact, if only one path consisted of one edge, from the considerations above, the tails of the other two would lay both on the same flat. Further, if only two

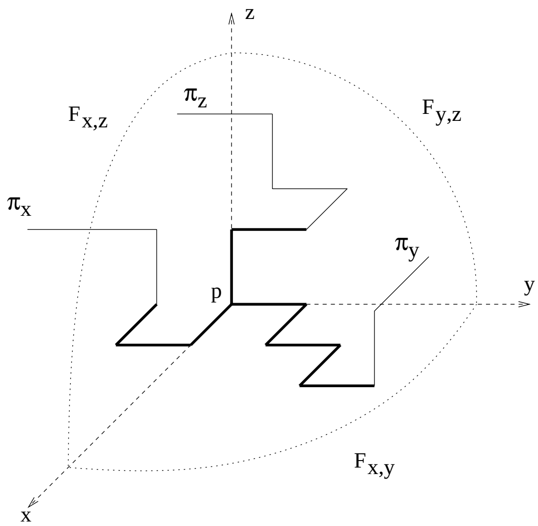

Figure 7: Three paths π_x, π_y, and π_z starting from a common point p. Paths π_x and π_y have their tails on flat $F_{x,y}$, while π_z has its tail on flat $F_{y,z}$. Tails are highlighted.

paths consisted of exactly one edge, then the third one would be on the same plane with one of them.

If $m_i = 1$, for all $i = x, y, z$, then the statement is trivially true. Suppose $m_i > 1$, for all $i = x, y, z$. We draw separately each path on a different plane and then join the three drawings. The length of the first segment $e_{i,1}$, with $i = x, y, z$, is one. Each path π_i is drawn independently by adding edge $e_{i,j}$, with $j = 2, \ldots, m_i$, in such a way that its end point is outside the bounding box of $e_{i,t}$, with $t = 1, \ldots, j-1$ and $e_{l,1}$, where π_l is the path with its first edge on the plane where π_i will be drawn. By construction, two edges belonging to the same path do not intersect. No intersection occurs between $e_{x,1}$, $e_{y,1}$, and $e_{z,1}$, since they are adjacent and orthogonal. No intersection occurs between $e_{i,1}$ and $e_{j,h}$, with $i \neq j$ and $h > 1$, since by construction $e_{j,h}$ is outside the bounding box of $e_{i,1}$. Intersections between $e_{i,h}$ and $e_{j,k}$, with $i \neq j$ and $h, k > 1$ are removable since $e_{i,h}$ and $e_{j,k}$ do not share a flat. □

Lemma 7 *Let π_x, π_y, and π_z be three shape paths starting from a common point p, such that: (i) $e_{x,1}$, $e_{y,1}$, and $e_{z,1}$ are pairwise orthogonal, (ii) at least one path among π_x, π_y, and π_z contains more than one flat and, (iii) at least one among m_x, m_y, and m_z is equal to one. If the paths $\overline{\pi_x} \cdot \pi_y$, $\overline{\pi_x} \cdot \pi_z$, and $\overline{\pi_y} \cdot \pi_z$ consist of exactly two edges or contain at least two flats, then π_x, π_y, and π_z admit a triply expanding drawing.*

Proof: Assume, without loss of generality, that π_x has more than one flat and that $m_z = 1$.

Since $\overline{\pi_x} \cdot \pi_z$ contains at least two flats, by Theorem 3 it is possible to build a double expanding drawing Π_{xz} of it. Denote by B_{xz} the bounding box of Π_{xz}, and let L_{max} be the maximum length of a side of B_{xz}.

Analogously, consider the drawing Π_{yz} of $\overline{\pi}_y \cdot \pi_z$ obtained as described in the proof of Theorem 3. Choose p as the origin and scale Π_{yz} by multiplying each coordinate by $2L_{max}$. Add the obtained drawing to Π_{xz} in such a way that the two drawings of point p coincide, and the two drawings of edge e_{z,m_z} overlap. We claim that the obtained drawing is a triple expanding drawing. In fact, both the drawings Π_{xz} and Π_{yz} are not self-intersecting because of Theorem 3, and the latter does not intersect B_{xz} because of the scaling.

Finally, intersections between edge e_{x,m_x} and $e_{y,k}$, with $k = 1, \ldots, m_i$, are removable since they involve edges that do not share a flat (consider that π_x contains more than one flat). $\square$

Lemma 8 *Let π_x, π_y, and π_z be three shape paths starting from a common point p, such that: (i) $e_{x,1}$, $e_{y,1}$, and $e_{z,1}$ are pairwise orthogonal, (ii) at least one path among π_x, π_y, and π_z contains more than one flat, and (iii) m_x, m_y, and m_z are greater than one. If the paths $\overline{\pi_x} \cdot \pi_y$, $\overline{\pi_x} \cdot \pi_z$, and $\overline{\pi_y} \cdot \pi_z$ contain at least two flats, then π_x, π_y, and π_z admit a triply expanding drawing.*

Proof: We prove the statement by construction: first we draw a pair of paths, and then we add the third one to the drawing. We rename paths π_x, π_y, and π_z in π_1, π_2, and π_3, where the index gives the order in which the paths will be drawn.

Two are the cases:

1. There exists a flat $F_{i,j}$ which hosts both the tails of π_i and π_j. In this case, we set $\pi_1 = \pi_i$, $\pi_2 = \pi_j$, and π_3 equal to the remaining path. Note that the tail of π_3 lies on flat $F_{h,3}$ (with h equal to 1 or 2) on which π_h has the edge $e_{h,1}$ only. See Figure 8.a for an example.

2. Such flat does not exist. In this case, we choose π_1 as any path that has more than one flat (there exists at least one by hypothesis), we choose π_2 in such a way that the tail of π_2 is on the flat $F_{1,2}$. The remaining path is π_3. Note that the tail of π_3 lies on a flat $F_{3,2}$ where π_2 has the edge $e_{2,1}$ only. See Figure 8.b for an example.

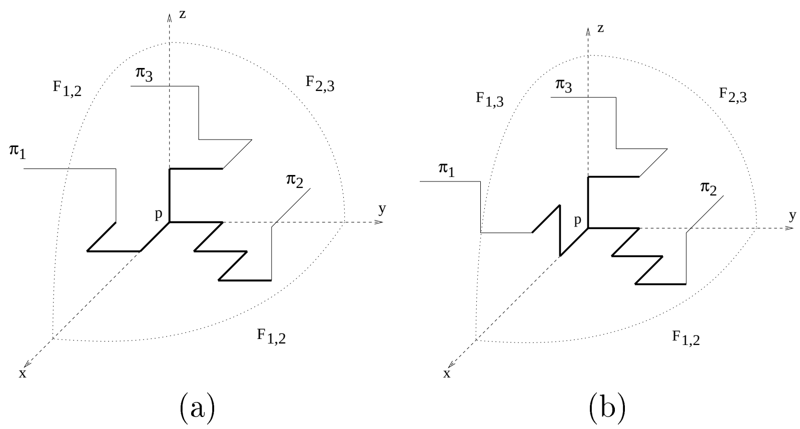

Figure 8: Case 1 (a) and Case 2 (b) of Lemma 8.

Since by hypothesis $\overline{\pi_1} \cdot \pi_2$ contains at least two flats, by Theorem 3 it admits a doubly expanding drawing. We realize such drawing as described in the proof of Theorem 3.

Starting from point p we add path π_3 to the construction. First, we show that, in both case 1 and case 2, edge $e_{3,1}$ can be drawn arbitrarily long without creating intersections on flats $F_{2,3}$ and $F_{1,3}$ (other intersections are removable since involve edges that do not share a flat).

In case 1 the only edges of $\overline{\pi_1} \cdot \pi_2$ on $F_{2,3}$ and $F_{1,3}$ are $e_{2,1}$ and $e_{1,1}$, respectively. Since they are adjacent and orthogonal to $e_{3,1}$, they can not cross it.

In case 2, $e_{3,1}$ does not intersect π_2, since on flat $F_{2,3}$ π_2 has the edge $e_{2,1}$ only, which is adjacent and orthogonal to $e_{3,1}$. Further, $e_{3,1}$ does not intersect π_1. In fact, let B_1 be the bounding box of π_1 deprived of $e_{1,1}$ and e_{1,m_1} (See Figure 9). Since π_1 is drawn as described by Theorem 3, applied on $\overline{\pi_1} \cdot \pi_2$, and since the way we choose π_1 in case 2 guarantees that there is a transition point between two flats in π_1 before reaching p, edge $e_{1,1}$ protrudes out of B_1. Since $e_{3,1}$ is outside B_1 its intersections with π_1 may occur with $e_{1,1}$ or e_{1,m_1} only. Edge $e_{3,1}$ is orthogonal to $e_{1,1}$ and adjacent to it. Thus there is no intersection between $e_{3,1}$ and $e_{1,1}$. The intersection that may occur between $e_{3,1}$ and e_{1,m_1} involves edges that do not share a flat (recall that π_1 contains at least two flats). Thus, $e_{3,1}$ can not intersect any edge of π_1.

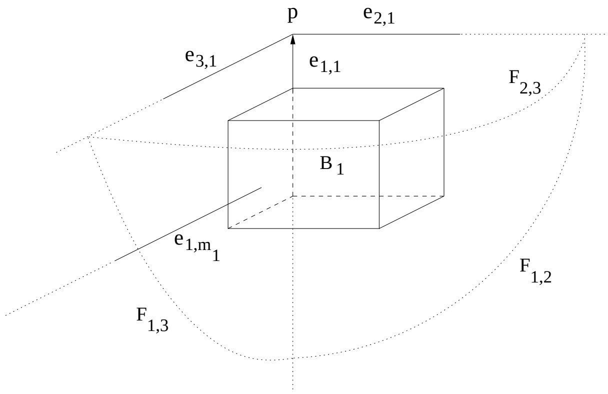

Figure 9: Edge $e_{3,1}$ does not cross edges e_{1,m_1} and $e_{2,1}$.

We now add edges $e_{3,i}$ for $i = 1, \ldots, m_3$ to the drawing. Denote by B_i, with $i = 1, \ldots, m_3$, the bounding box of the current drawing before the insertion of edge $e_{3,i}$ deprived of e_{1,m_1} and e_{2,m_2}. We add each edge $e_{3,i}$ for $i = 1, \ldots, m_3$ in such a way that its end point is placed one unit ouside B_i. An intersection may occur only with edges e_{1,m_1} and e_{2,m_2}, which are the only two edges protruding from B_i when edge $e_{3,i}$ is added. Such intersections involve edges that do not share a flat.

Thus, π_3 may be added to the drawing without intersecting the drawing of $\overline{\pi_1} \cdot \pi_2$, and in such a way that the last added edge e_{3,m_3} could be drawn arbitrarily long. $\square$

Lemma 9 *Let π_x, π_y, and π_z be three shape paths starting from a common point p, such that $e_{x,1}$ and $e_{y,1}$ have opposite direction labels. If the paths $\overline{\pi_x} \cdot \pi_y$, $\overline{\pi_x} \cdot \pi_z$, and $\overline{\pi_y} \cdot \pi_z$ consist of exactly two edges or contain at least two flats, then π_x, π_y, and π_z admit a triply expanding drawing.*

Proof: By hypothesis $\overline{\pi_x} \cdot \pi_y$ consists of exactly two edges or contains at least two flats. In the latter case, there exists an edge in $\overline{\pi_x} \cdot \pi_y$ orthogonal to both $e_{x,1}$ and $e_{z,1}$. Suppose, without loss of generality, that such edge is in π_x, and that k is the smallest index for which $e_{x,k}$ is orthogonal to both $e_{x,1}$ and $e_{z,1}$. Call X_i the direction label of $e_{x,i}$.

Denote by γ_0 the shape graph composed by segments $e_{x,j}$, with $j = k, \ldots, m_x$, and paths π_y and π_z. Denote by γ_i, with $i = 1, \ldots, k-1$, the shape graph composed by segments $e_{x,j}$, with $j = k, \ldots, m_x$, segments $e_{x,h}$, with $h = 1, \ldots, i$, and paths π_y, and π_z. Note that γ_{k-1} coincides with the shape graph composed by π_x, π_y, and π_z.

By Lemma 6, Lemma 7, and Lemma 8, γ_0 admits a triply expanding drawing. We will iteratively apply Lemma 3 in order to obtain 3D simple orthogonal drawings of γ_i, with $i = 1, \ldots, k-1$. At iteration 1 $e_{x,1}$ can be inserted between p and $e_{x,k}$, since X_1 is different from the direction labels of $e_{y,1}$, $e_{z,1}$, and $e_{x,k}$. At iteration i, with $i = 2, \ldots, k-1$, $e_{x,i}$ can be inserted between $e_{x,i-1}$ and $e_{x,k}$, since X_i is orthogonal both to X_{i-1} and to X_k.

If $\overline{\pi_x} \cdot \pi_y$ consists of exactly two edges, consider the doubly expanding drawing of $\overline{\pi_z} \cdot \pi_x$ constructed as described in the proof of Theorem 3. Observe that such drawing differs from the drawing of $\overline{\pi_z} \cdot \pi_y$ in the last added edge ($e_{x,1}$ instead of $e_{y,1}$) only. Thus, the triply expanding drawing may be obtained by adding $e_{y,1}$ to the doubly expanding drawing of $\overline{\pi_z} \cdot \pi_x$. □

From Lemmas 6, 7, 8, and 9 follows:

Theorem 4 *Let π_x, π_y, and π_z be three shape paths starting from a common point p. If the paths $\overline{\pi_x} \cdot \pi_y$, $\overline{\pi_x} \cdot \pi_z$, and $\overline{\pi_y} \cdot \pi_z$ consist of exactly two edges or contain at least two flats, then π_x, π_y, and π_z admit a triply expanding drawing.*

5 A Sufficient Condition for Theta Shapes Simplicity

Let Θ be a theta shape composed by three shape paths π_1, π_2, and π_3 from point p to point q. Let $e_{i,j}$, with $j = 1, \ldots, m_i$, the j-th edge of π_i. In the following we denote by $l_{i,j}$ the label associated with $e_{i,j}$ when directed according to π_i. Recall that when π_i is reversed, as in $C_{k,i} = \pi_k \cdot \overline{\pi_i}$, the label associated with $e_{i,j}$ is the opposite of $l_{i,j}$, that is $\overline{l_{i,j}}$.

Theorem 5 *Let Θ be a theta shape composed by three shape paths π_1, π_2, and π_3 from point p to point q. If for each π_i there exist three edges $e_{i,1}$, $e_{i,2}$, and $e_{i,3}$, such that for each pair of paths π_i and π_j, $i, j = 1, 2, 3$, $i \neq j$, the six labels $l_{i,1}$, $l_{i,2}$, $l_{i,3}$, $\overline{l_{j,1}}$, $\overline{l_{j,2}}$, and $\overline{l_{j,3}}$ form a canonical sequence $\tau_{i,j}$ for the shape cycle $C_{i,j} = \pi_i \cdot \overline{\pi_j}$, then Θ is simple.*

Proof:
The following properties hold for labels $l_{i,j}$:

1. The labels $l_{i,j}$ of the same path are different, i.e., $l_{i,j} \neq l_{i,k}$, $i, j, k = 1, 2, 3$, $j \neq k$.
2. No two labels $l_{i,j}$ of the same path are opposite, i.e., $l_{i,j} \neq \overline{l_{i,k}}$, $i, j, k = 1, 2, 3$, $j \neq k$.
3. No two labels $l_{i,j}$ of different paths are opposite, i.e., $l_{i,j} \neq \overline{l_{h,k}}$, $i, j, h, k = 1, 2, 3$, $i \neq h$.

Property 1 follows from the canonicity of $\tau_{1,2}$, $\tau_{1,3}$, and $\tau_{2,3}$. Property 2 can be proved by contradiction. In fact, suppose that $l_{i,j} = \overline{l_{i,k}}$ for some i, j, k with $j \neq k$. Consider $C_{i,m}$ $m \neq i$. From the canonicity of $\tau_{i,m}$ it follows that $l_{m,h} \neq l_{i,j}$ and $l_{m,h} \neq \overline{l_{i,k}}$, $h = 1, 2, 3$. It follows that $\tau_{m,n}$, $m, n \neq i$ contains neither the direction of $l_{i,j}$ nor its opposite, contradicting its canonicity. Property 3 can be also proved by contradiction. Suppose that $l_{i,j} = \overline{l_{h,k}}$, $i \neq h$ and consider $C_{i,h} = \pi_i \cdot \overline{\pi_h}$. Since π_h is reversed, the direction of label $l_{i,j}$ occurs twice in $\tau_{i,h}$.

From Properties 1, 2, and 3 it follows that, for each $i = 1, 2, 3$, labels $l_{i,1}$ $l_{i,2}$ $l_{i,3}$ are a permutation of the same three labels, one for each oppositely directed pair. We assume, without loss of generality, that $l_{i,j} \in \{U, N, E\}$ with $i, j = 1, 2, 3$.

The nine edges $e_{i,j}$, $i, j = 1, 2, 3$, bound eight connected subgraphs of the theta graph. We call $G_{i,j}$ the subpath of π_i from $e_{i,j}$ to $e_{i,j+1}$, with $i = 1, 2, 3$, and $j = 1, 2$. Further, we call G_p the subgraph composed by the three paths from p to $e_{i,1}$ with $i = 1, 2, 3$, and G_q the subgraph composed by the three paths from $e_{i,3}$ with $i = 1, 2, 3$ to q.

We draw separately each one of such subgraphs, then we decide the length of the segments $e_{i,j}$, that are common to the drawings of two subgraphs, in such a way that no unremovable intersection occurs in the drawing.

Observe that $G_{i,j}$, with $i = 1, 2, 3$, and $j = 1, 2$ admits a doubly expanding drawing. In fact, since the labels associated with $e_{i,j}$ and $e_{i,j+1}$ are canonical, it follows that, if they are not consecutive, they do not share a flat, and Theorem 3 applies. Also, observe that G_p admits a triply expanding drawing. In fact, since, for each pair i, j, $e_{i,1}$ and $e_{j,1}$ are canonical in $C_{i,j}$, it follows that, if they are not consecutive, they do not share a flat, and Theorem 4 applies. Analogously, G_q admits a triply expanding drawing.

We draw a doubly expanding drawing $\Gamma_{i,j}$ for each $G_{i,j}$, with $i = 1, 2, 3$, and $j = 1, 2$ as described in the proof of Theorem 3. Let L be the maximum length of a side of a bounding box of the drawings obtained above. We draw a triply expanding drawing Γ_p of G_p and a triply expanding drawing Γ_q of G_q as described in Theorem 4.

Now we show how to set the lengths $\lambda_{i,j}$, $i, j = 1, 2, 3$, of the part of each edge $e_{i,j}$ that is not contained in the bounding box of any expanding drawing, since the actual lengths of the edges can be easily computed from it.

Denote by B_p, B_q, and $B_{i,j}$, with $i = 1, 2, 3$ and $j = 1, 2$, the bounding boxes of Γ_p, Γ_q, and $\Gamma_{i,j}$, respectively. Since edges $e_{i,1}$, with $i = 1, 2, 3$, have labels in $\{U, N, E\}$, there are three pairwise orthogonal sides of B_p from which edges $e_{i,1}$ may come out. Denote by v_{B_p} the only point common to these sides. Denote by v_{B_q} the point common to the analogous sides of B_q. Place the drawings Γ_p and Γ_q in such a way that v_{B_p} has coordinates $(-2L - 1, -2L - 1, -2L - 1)$ and v_{B_q} has coordinates $(2L + 1, 2L + 1, 2L + 1)$.

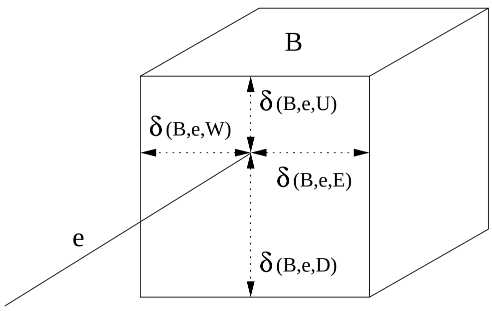

Figure 10: The definition of $\delta(B, e, X)$.

Now we show how to add the drawing of $\Gamma_{i,1}$ and $\Gamma_{i,2}$, for $i = 1, 2, 3$. Given a bounding box B, an edge e protruding from it, and a direction label X orthogonal to e, we denote by $\delta(B, e, X)$ the distance of e from the side of B in the X direction (see Figure 10). We assign to $\lambda_{i,j}$, with $i, j = 1, 2, 3$, the following values.

$$\lambda_{i,1} = 4L + 2 + \delta(B_q, e_{i,3}, \overline{l_{i,1}}) - \delta(B_{i,2}, e_{i,3}, \overline{l_{i,1}}) + \delta(B_{i,2}, e_{i,2}, \overline{l_{i,1}}) - \delta(B_{i,1}, e_{i,2}, \overline{l_{i,1}}) \quad (1)$$

$$\lambda_{i,2} = 4L + 2 + \delta(B_p, e_{i,1}, l_{i,2}) + \delta(B_q, e_{i,3}, \overline{l_{i,2}}) - \delta(B_{i,1}, e_{i,1}, l_{i,2}) - \delta(B_{i,2}, e_{i,3}, \overline{l_{i,2}}) \quad (2)$$

$$\lambda_{i,3} = 4L + 2 + \delta(B_p, e_{i,1}, l_{i,3}) - \delta(B_{i,1}, e_{i,1}, l_{i,3}) + \delta(B_{i,1}, e_{i,2}, l_{i,3}) - \delta(B_{i,2}, e_{i,2}, l_{i,3}) \quad (3)$$

Now we show that no intersection between edges sharing a flat has been introduced in the drawing. By Theorems 3 and 4 no intersection occurs in each (doubly or triply) expanding drawing.

There is no intersection between two edges of the same π_i. In fact, since $0 \leq \delta(B_{i,j}, e, X) \leq L$, and by equations 1, 2, and 3, each $\lambda_{i,j}$, with $i, j = 1, 2, 3$ is at least $2L+2$, each one of B_p, B_q, $B_{i,1}$, and $B_{i,2}$ is in a different octant.

Intersections between edges belonging to different paths involve edges that do not share a flat, and thus by Lemma 5 can be removed. Indeed, suppose for a contradiction that e_x and e_y are two intersecting edges belonging to two different paths π_x and π_y and that e_x and e_y are on the same flat F. Edges e_x and e_y do not belong to the same expanding drawing, since otherwise they could not intersect.

There must exist a path π^* joining e_x and e_y, entirely contained in F, and containing p or q. In π^* there can be at most three of the edges $e_{i,j}$, with $i = x, y$, and $j = 1, 2, 3$, since otherwise there would be more than three canonical labels on the same flat for the cycle $C_{x,y}$, contradicting the hypothesis that $\tau_{x,y}$ is a canonical sequence. Since edges $e_{i,j}$ in π^* are on the same flat F and are canonical, they are necessarily adjacent, and thus if they are consecutive, they are orthogonal with each other. If e_x and e_y coincide with two edges $e_{i,j}$ then they can not intersect since they are parallel or adjacent and orthogonal. Otherwise, since edges $e_{i,j}$ are the only ones to transition between octants then e_x and e_y belong to two different octants and they can not intersect. □

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