UNIVERSITÀ DEGLI STUDI DI ROMA TRE
Dipartimento di Informatica e Automazione
Via della Vasca Navale, 79 – 00146 Roma, Italy

Polynomial Area Bounds for MST Embeddings of Trees

FABRIZIO FRATI[†] AND MICHAEL KAUFMANN[‡]

RT-DIA-122-2008 **Agosto 2008**

[†] Dipartimento di Informatica e Automazione – Università di Roma Tre, Italy.
[‡] Wilhelm-Schickard-Institut für Informatik – Universität Tübingen, Germany.

ABSTRACT

In their seminal paper on geometric minimum spanning trees, Monma and Suri [11] showed how to embed any tree of maximum degree 5 as a minimum spanning tree in the Euclidean plane. The embeddings provided by their algorithm require area $O(2^{n^2}) \times O(2^{n^2})$ and the authors conjectured that an improvement below $c^n \times c^n$ is not possible, for some constant $c > 0$. In this paper, we show how to construct MST embeddings of arbitrary trees of maximum degree 3 and 4 within polynomial area.

1 Introduction

A *minimum spanning tree* (MST) of a set P of points in the plane is defined as a tree having a vertex for each point of P and having minimum total edge length. As the distance between any two non-adjacent vertices u and v of a minimum spanning tree T must be at least as large as the distance between any two vertices on the path between u and v in T, the MST reflects certain proximity relations in a set of points in the plane, playing important roles in various fields of computer science. For example, minimum spanning trees are widely used in the field of sensor networks, namely their topologies guarantee total connection between the nodes of a network, while minimizing the total energy consumption of the sensors (see, e.g., [4]).

Given a set P of n points in the plane, it is well-known that the minimum spanning tree of P can be computed in optimal $\Theta(n \log n)$ time, however the computation of a minimum spanning tree subject to further constraints is often required. The boundedness of the degree of the nodes of the tree is a natural constraint to consider, since having high-degree nodes is in many ways undesiderable. It is well-known that every set of points in the plane has a minimum spanning tree with maximum degree 5 [11]. If the maximum degree of the nodes is constrained to be bounded by 2 or 3, then computing a minimum spanning tree is $\mathcal{NP}$-hard [6, 12] (the complexity status of the same problem is still unknown if the degree of the tree is bounded by 4). However, a polynomial-time approximation scheme is known if the maximum degree of the tree is required to be at most 2 [1, 10], an $O(n^{\log^c n})$-time $(1+\epsilon)$-approximation algorithm [2] and a polynomial-time 1.402-approximation algorithm [3] are known if the maximum degree of the tree is required to be at most 3, and a polynomial-time 1.143-approximation algorithm [3] is known if the maximum degree of the tree is required to be at most 4.

Consider a tree T. Does T admit an *MST embedding*, i.e., a straight-line drawing in which the minimum spanning tree of the points where the vertices of T are placed at coincides with T? Such a question is in the following regarded as the *MST embedding problem*. Monma and Suri [11] provided an algorithm to construct an MST embedding of any tree of maximum degree 5 and proved that any tree having a node of degree at least 7 does not admit an MST embedding. Eades and Whitesides [5] filled the gap in Monma and Suri's results, by proving that deciding whether an MST embedding exists for a given tree of maximum degree 6 is $\mathcal{NP}$-hard.

Extensions to higher dimensions have been performed by Di Battista and Liotta [9], as well as by King [7]. In the former paper, the authors proved that trees with maximum degree 9 can be embedded as MSTs in the three-dimensional Euclidean space; in the latter paper, it is proved that every tree of maximal degree 10 admits an MST embedding in three dimensions. It is also known that no tree having a vertex of degree at least 13 admits an MST embedding in three dimensions [8].

Monma and Suri's proof that every tree of maximum degree five admits an MST embedding in the plane is a strong combinatorial result. However, their algorithm for constructing MST embeddings of trees turns out to be useless in practice, since the constructed drawings require an area of $O(2^{k^2}) \times O(2^{k^2})$ for trees of height k (hence, in the worst case the area requirement of the drawings is doubly-exponential in the number of nodes of the tree). Notice that the algorithm of Monma and Suri does not give a polynomial area bound even for *complete binary trees*, namely the algorithm provides an $O(n^{\log n})$ area bound in such a case. However, Monma and Suri conjectured that there exist trees of maximum degree 5 that require $c^n \times c^n$ area in *any* MST embedding, for some constant $c > 1$. The problem of determining whether or not the area upper bound for MST embeddings of trees can be improved to polynomial is reported also in [5].

In this paper, we concentrate on the area requirements for MST embeddings of trees in the plane. In particular we derive polynomial area bounds for MST embeddings of trees with maximum degree 3 and 4. Some attention is devoted to complete trees of degree 3 and 4, for which we show simple algorithms to construct MST embeddings within small area.

The rest of the paper is organized as follows: Section 2 contains some preliminaries, Sections 3, 4, 5, and 6 show how to construct MST embeddings of complete binary trees, of arbitrary binary trees, of complete ternary trees, and of arbitrary ternary trees, respectively, and Section 7 presents some conclusions.

We notice that a polynomial area bound for arbitrary ternary trees implies polynomial area bounds for complete binary trees, for arbitrary binary trees, and for complete ternary trees, that are all subclasses of arbitrary ternary trees. However, we still present algorithms for constructing MST embeddings of complete binary trees, of arbitrary binary trees, and of complete ternary trees, motivated both by the simplicity of the corresponding algorithms, and by the better area bounds that we can achieve in such cases. Notice also that we do not strive for the best polynomial bounds but try to keep the techniques and the analysis as simple as possible. Nevertheless, we achieve the first polynomial area bounds drastically improving from the previous exponential ones.

2 Preliminaries

We introduce the basic notations although some have been mentioned before in the introduction. A *tree* is a connected acyclic graph. The *degree of a node* is the number of edges incident to it. The *degree of a tree* is the maximum degree of one of its nodes. A *rooted tree* is a tree with one distinguished node, called *root*. *Binary trees* and *ternary trees* are trees of maximum degree 3 and 4, respectively, that are rooted at any node of degree at most 2 and 3, respectively. In a rooted binary tree (resp. ternary tree), each node has at most 2 *children* (resp. 3 children), and a *leaf* is a node without children. The *height* of a rooted tree is the length of the longest path from the root to a leaf. A complete tree is such that each non-leaf node has the same number of children, and all paths from the root to a leaf have the same number of nodes.

A *straight-line drawing* of a tree is a mapping of each node to a point in the plane and of each edge to a straight-line segment between its endpoints.

A *minimum spanning tree* MST of a set of n points in the plane is defined to be a tree spanning the n points and having minimum total cost, where the cost of each edge (u, v) is defined as the Euclidean distance between u and v. Given a tree T, the *MST embedding problem* asks for a mapping of the vertices of T to points in the plane such that the minimum spanning tree of such points is the input tree T. Such a mapping provides a straight-line drawing of T, that is called an *MST embedding* of T. A necessary and sufficient condition is known for a straight-line drawing of a tree T to be an MST embedding of T.

Property 1 *A straight-line drawing Γ of a tree T is an MST embedding of T if and only if, for each pair of non-adjacent nodes u and v of T, their Euclidean distance in Γ is greater or equal than the length of each edge in the path connecting u and v in T.*

Given a straight-line drawing Γ of a tree T, we call *MST condition* the necessary and sufficient condition for Γ to be an MST embedding of T expressed by the previous property. In the following, we will show algorithms for constructing MST embeddings of trees and we will prove that the constructed straight-line drawings are MST embeddings by verifying that, for each pair of non-adjacent nodes of a tree, their distance is at least the length of each edge in the path connecting them.

The *area* of a straight-line drawing is the area of a rectangle enclosing such a drawing. Notice that the concept of area of a drawing of a graph only makes sense once fixed a *resolution rule*, i.e., a rule that does not allow vertices to be arbitrarily close (*vertex resolution rule*), or edges to be arbitrarily short (*edge resolution rule*). In fact, without any of such rules, one could just construct arbitrarily small drawings and enclose them in an arbitrarily small area. In the following we will only refer to the edge resolution rule, hence we will have to ensure that the shortest edge of the drawing has length at least one unit. This is not a drawback of our algorithm, since for MST embeddings of trees the edge resolution rule implies the vertex resolution rule. Namely, two adjacent vertices cannot be closer than one unit distance, by the edge resolution rule. Further, two non-adjacent vertices cannot be closer than one unit distance, otherwise, by the MST condition, there would be an edge in the path connecting such two vertices shorter than one unit distance, again contradicting the edge resolution rule.

3 MST Embeddings of Complete Binary Trees

In this section we deal with the construction of MST embeddings of complete binary trees.

Let T be a complete binary tree with n nodes and let $n = 2^k - 1$, for some integer k. Tree T consists of a root r and of two subtrees T_1 and T_2 rooted at the children r_1 and r_2 of r, respectively. Each of T_1 and T_2 has size less than $n/2$. We recursively embed T_1 and T_2 into two equal isosceles right triangles Δ_1 and Δ_2, respectively, so that the root of a subtree T_i is placed at the vertex of Δ_i incident to the 90-degree angle.

When T has only one node, such a node is placed at the vertex incident to the 90 degrees angle of an isosceles right triangle Δ having sides of length one.

When T has more than one node, we place Δ_1 and Δ_2 with their hypotenuses on the same horizontal line, at distance d from each other, where d is a value that will be chosen later. Let L denote the length of a side of Δ_1 and Δ_2. We place r at the intersection of the perpendicular lines on which a side of Δ_1 and a side of Δ_2 lie. The whole drawing is contained inside an isosceles right triangle Δ having sides of length $(c+1)L$, where c is a constant that will be determined later. Observe that r is placed at the vertex of Δ incident to the 90-degree angle. See Fig. 1.

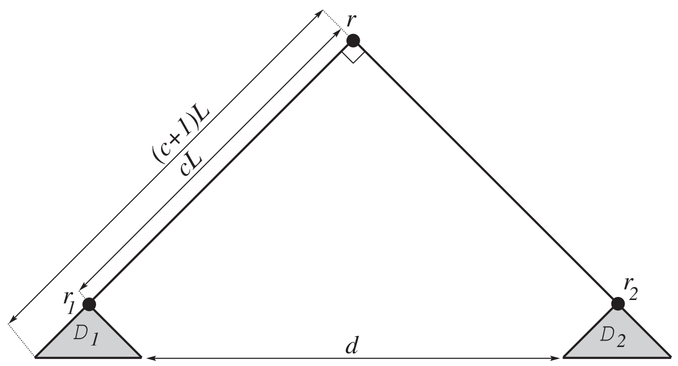

Figure 1: The recursive construction of an MST embedding of a complete binary tree.

We prove that the constructed drawing is an MST embedding of T, for some value of c. Inductively assume that the drawings of subtrees T_1 and T_2 are MST embeddings. Then, we have only to prove that each straight-line segment connecting a node w_1 in T_1 and a node w_2 in T_2 is longer than each edge of the path connecting w_1 and w_2 in T. By construction, the distance between w_1 and w_2 is at least d. The edges belonging to the path connecting w_1 and w_2 in T have length at most $\max\{\sqrt{2}L, cL\}$, namely all such edges are contained inside Δ_1 and Δ_2, but for (r, r_1) and (r, r_2), that by construction have length cL. Observe that, by construction, $d = \sqrt{2}(c-1)L$. Hence, as long as $c \geq \sqrt{2}/(\sqrt{2}-1)$, d is greater or equal than both cL and $\sqrt{2}L$, so the constructed drawing is an MST embedding of T.

We now compute the area of the constructed drawing, which is bounded by the area of Δ. Observe that each edge of the drawing has length at least one. Denote by $S(n)$ the length of the side of Δ, when the input is a complete binary tree with n nodes. We get: $S(n) = (c+1)S(\frac{n}{2}) = \left(\frac{2\sqrt{2}-1}{\sqrt{2}-1}\right)^{\log_2 n} = n^{\log_2 \frac{2\sqrt{2}-1}{\sqrt{2}-1}} \leq n^{\log_2 4.415} \leq n^{2.15}$. Since the area of Δ is asymptotically the square of its side, we obtain the following:

Theorem 1 *A complete binary tree with n vertices admits an MST embedding in $O(n^{4.3})$ area.*

4 MST Embeddings of Arbitrary Binary Trees

Now we present an algorithm to construct MST embeddings of arbitrary binary trees.

Overall strategy. Assume that the input binary tree T is rooted at any node r of degree at most two. Select a *chain* $P = (r = v_1, v_2, v_3, \cdots, v_k)$ in T, that is, a path from r to a leaf. Remove the chain from the tree, disconnecting the tree into several subtrees. Recursively draw the disconnected subtrees and place a drawing of the chain together with the drawings of the subtrees, obtaining a drawing of the whole tree.

Choice of the chain. The choice of P is done as follows. The first node v_1 of P is r. For each $1 \leq i < k$, node v_{i+1} is defined as the root of the larger of the two subtrees of v_i. Observe that each subtree of P has at most $n/2$ nodes.

The shape of the subtrees. Denote by T_i the subtree rooted at the child t_i of v_i that does not belong to P. We recursively draw the subtrees T_i of P inside isosceles right triangles Δ_i, for $1 \leq i \leq k-1$. The whole chain together with the drawing of the subtrees of P will be placed inside a larger isosceles right triangle Δ. The root of each subtree T_i is placed on the midpoint of the hypotenuse of Δ_i. Denote by L_i the length of the hypotenuse of Δ_i.

Drawing the chain and the subtrees together. Let $e_i = (v_i, v_{i+1})$, for $1 \leq i < k$. We draw P in a *zig-zag* way, with constant angles of 120 degrees between two consecutive edges e_i and e_{i+1}. The length of edges e_i will be determined later.

Consider vertex v_i. Opposite to the 120 degree angle, we have an angle of 240 degrees, which we partition into four consecutive wedges W_i^1, W_i^2, W_i^3, and W_i^4 of 90, 30, 30, and 90 degrees, respectively, such that W_i^1 is the wedge closer to vertex v_{i-1}. See Fig. 2. We place Δ_i inside W_i^3 as follows. Consider the line l_i through v_i bisecting W_i^3. Vertex t_i is placed on l_i and triangle Δ_i is placed inside W_i^3 so that the hypotenuse of Δ_i is perpendicular to l_i, and so that the endvertices of the hypotenuse of Δ_i lie on the semi-axes delimiting W_i^3. See Fig. 3.

Notice that, for vertex v_1 (and for vertex v_k), wedges W_i^1, W_i^2, W_i^3, and W_i^4 are not well-defined, since only one edge e_1 of P is incident to v_1. However, it is not difficult to extend the above definition of wedges W_i^1, W_i^2, W_i^3, and W_i^4 to the case in which $i = 1$, by considering a dummy edge (v_0, v_1) that has an angle of 120 degrees with edge (v_1, v_2), and defining the wedges incident to v_1 as for the other vertices of P.

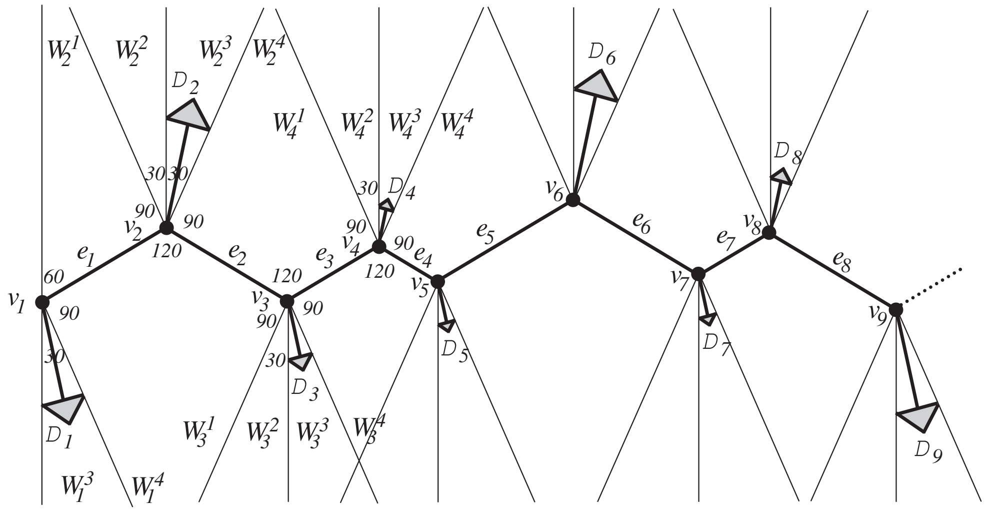

Figure 2: The recursive construction of an MST embedding of an arbitrary binary tree.

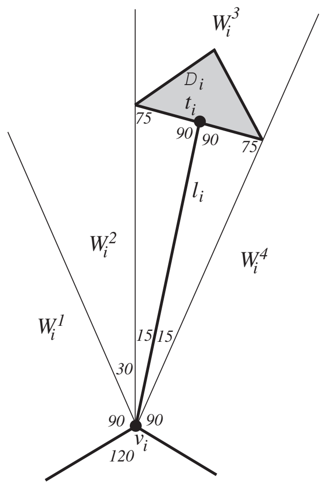

Figure 3: A closer look to the construction of an MST embedding of an arbitrary binary tree.

Choosing the length of edges e_i. We set:

$$len(e_i) = max\{cL_i, cL_{i+1}\},$$

where c is a constant greater than one to be determined later. In order to have positive lengths for all edges, we set $len(e_i) = 1$, for all edges e_i where none of subtrees T_i and T_{i+1} exists.

The isosceles right triangle Δ is defined as the smallest isosceles right triangle containing the whole drawing, having r as midpoint of the hypotenuse, and having the hypotenuse forming angles of 120, 60, and 180 with edge (v_1, v_2). In the following we suppose, for clarity of exposition, that the hypotenuse of Δ is vertical, and that P is contained in the half-plane to the right of the line through the hypotenuse. If a subtree T_i has only one node, Δ is defined as the isosceles right triangle having r as midpoint of the hypotenuse, and having the hypotenuse such that $L_i = 1$.

The drawing satisfies the MST condition. We use induction to show that every pair of vertices in the drawing satisfies the MST condition. If the tree has only one node, then there is nothing to prove. Otherwise, inductively suppose that each pair of nodes in the drawing of each subtree T_i satisfies the MST condition. Then, we prove that each pair of nodes in the whole drawing satisfies the MST condition.

The only pairs of nodes for which the MST condition is not trivially satisfied, are: (i) node v_i and any node in T_i, for $i = 1, 2, \cdots, k-1$, (ii) node v_i and any node in T_{i-1}, for $i = 2, 3, \cdots, k$, (iii) node v_i and any node in T_{i+1}, for $i = 1, 2, \cdots, k-2$, and (iv) any node of $T_{i-1} \cup \{v_{i-1}\}$ and any node in $T_{i+1} \cup \{v_{i+1}\}$, for $i = 2, 3, \cdots, k-2$.

(i) Consider node v_i and any node w_i in T_i, for any $i = 1, 2, \cdots, k-1$. We prove that all edges in the path from v_i to w_i are shorter than segment $\overline{v_i w_i}$. Each edge of such a path belonging to T_i has length at most L_i. The length of edge (v_i, t_i) is equal to $L_i/(2 \cdot \tan(15)) \geq 1.866 L_i$. Hence, (v_i, t_i) is the longest edge of the path connecting v_i and w_i. However, $\overline{v_i, w_i}$ is longer than (v_i, t_i), since w_i is contained inside Δ_i, whose closest point to v_i is t_i.

(ii) For any $i = 2, 3, \cdots, k-1$, consider a node v_i and any node w_{i-1} in T_{i-1}, and suppose that the pair (v_i, w_{i-1}) of vertices does not satisfy the MST condition. As in the previous case each edge of such a path belonging also to T_{i-1} has length at most L_{i-1}. Further, edge (v_{i-1}, t_{i-1}) has length $L_i/(2 \cdot \tan(15)) \leq 1.867 L_i$, and edge (v_{i-1}, v_i) has length at least cL_{i-1}. It follows that, as long as $c \geq 1.867$, edge (v_{i-1}, v_i) is the longest edge in the path connecting v_i and w_{i-1}. However, consider triangle (v_i, v_{i-1}, w_{i-1}). By construction, angle $v_i \widehat{v}_{i-1} w_{i-1}$ contains wedge W_i^4 and hence it is greater or equal than 90 degrees. Segment $\overline{v_i w_i}$ is opposite to $v_i \widehat{v}_{i-1} w_{i-1}$ and hence is the longest side of such a triangle. It follows that $\overline{v_i w_i}$ is longer than (v_{i-1}, v_i).

(iii) For any $i = 1, 2, \cdots, k-2$, it can be proved analogously to the previous case that the MST of the points of the drawing cannot contain an edge (v_i, w_{i+1}), for any node w_{i+1} in T_{i+1}.

(iv) Consider any node w_{i-1} in $T_{i-1} \cup \{v_{i-1}\}$ and any node w_{i+1} in $T_{i+1} \cup \{v_{i+1}\}$, for $i = 2, 3, \cdots, k-2$. The path P_{i-1}^{i+1} connecting w_{i-1} and w_{i+1} in T contains edges e_{i-1} and e_i. All edges of P_{i-1}^{i+1} belonging to T_{i-1} or to T_{i+1} are contained inside Δ_{i-1} or Δ_{i+1}, respectively, and hence their length is at most the maximum between L_{i-1} and L_{i+1}. Further, the length of edge $\overline{v_{i-1} t_{i-1}}$ is $L_{i-1}/(2 \cdot \tan(15)) \leq 1.867 L_{i-1}$. Analogously, the length of edge $\overline{v_{i+1} t_{i+1}}$ is at most $1.867 L_{i+1}$. Hence, the length of each edge in P_{i-1}^{i+1} is less or equal than $\max\{1.867 L_{i-1}, 1.867 L_{i+1}, len(e_{i-1}), len(e_i)\}$. Observe that, by construction, $len(e_{i-1}) \geq cL_{i-1}$, and that $len(e_i) \geq cL_{i+1}$. Hence, as long as $c \geq 1.867$, one edge between e_i and e_{i+1} is the longest edge in P_{i-1}^{i+1}, and we have only to prove that the distance between w_{i-1} and w_{i+1} is greater than $\max\{len(e_{i-1}), len(e_i)\}$. In the following, refer to Fig. 4.

Consider line $l_{i-1}^{3,4}$ separating wedges W_{i-1}^3 and W_{i-1}^4, and consider line $l_i^{1,2}$ separating wedges W_i^1 and W_i^2. By construction such lines are parallel. Further, T_{i-1} is contained in the half-plane delimited by $l_{i-1}^{3,4}$ and not containing $l_i^{1,2}$. Notice that the distance between $l_{i-1}^{3,4}$ and $l_i^{1,2}$ is exactly $len(e_{i-1})$. We claim that, for a suitable constant c, T_{i+1} is entirely contained in the half-plane delimited by $l_i^{1,2}$ and not containing $l_{i-1}^{3,4}$. The claim clearly implies that the distance between w_{i-1} and w_{i+1} is greater or equal than $len(e_{i-1})$.

Let v_{i+1}^C be the vertex of Δ_{i+1} on the line l_{i+1}^C separating wedges W_{i+1}^2 and W_{i+1}^3. By construction, T_{i+1} entirely lies in the half-plane that is delimited by the line with slope 60 degrees through v_{i+1}^C and that does not contain $l_{i-1}^{3,4}$ and $l_i^{1,2}$. Hence, we have only to prove that, for a suitable constant c, v_{i+1}^C is in the half-plane delimited by $l_i^{1,2}$ and not containing $l_{i-1}^{3,4}$.

The vertical distance between v_{i+1} and v_{i+1}^C is easily computed to be $L_{i+1}/(2 \cdot \sin(15))$. The vertical distance between v_{i+1} and the intersection point u_i^C of l_i^C and $l_i^{1,2}$ is exactly $len(e_i)$, since triangle (v_i, v_{i+1}, u_i^C) is an isosceles triangle with catheti (v_i, v_{i+1}) and (v_{i+1}, u_i^C). It follows that $\overline{v_{i+1} u_i^C}$ is at least cL_{i+1}. Hence, v_{i+1}^C is in the half-plane delimited by $l_i^{1,2}$ and not

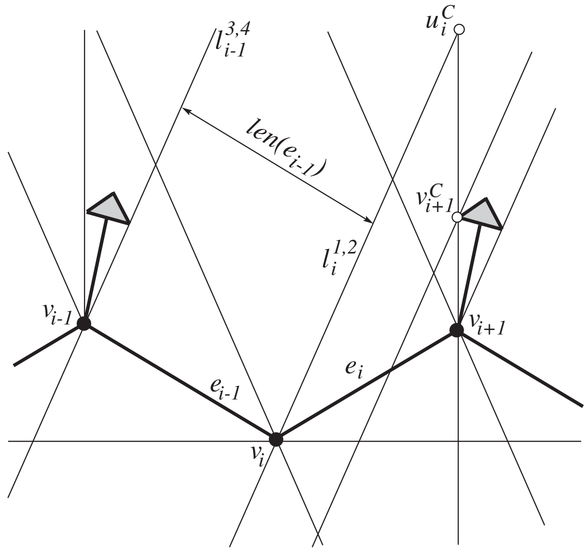

Figure 4: Illustration for the proof that the MST condition is satisfied for any node of $T_{i-1} \cup \{v_{i-1}\}$ and any node in $T_{i+1} \cup \{v_{i+1}\}$, for $i = 2, 3, \cdots, k-2$.

containing $l^{3,4}_{i-1}$ as long as $cL_{i+1} \geq L_{i+1}/(2 \cdot \sin(15))$, i.e., as long as $c \geq 1.932$.

In analogous way, it can be proved that, as long as $c \geq 1.932$, the distance between w_{i-1} and w_{i+1} is greater than $len(e_i)$. Hence, as long as $c \geq 1.932$, the straight-line segment between w_{i-1} and w_{i+1} is longer than every edge in the path P^{i+1}_{i-1} connecting w_{i-1} and w_{i+1} in T, and hence it does not belong to the MST of the points of the drawing.

The length of P. We bound the length of P as a function of the lengths L_i's. Since $len(e_i) = max\{cL_i, cL_{i+1}\}$, and since $len(e_i) \geq 1$, for every $1 \leq i < k$, then $len(e_i) < cL_i + cL_{i+1}$. It follows that $\sum_{i=1}^{k-1} len(e_i) \leq 2c \sum_{i=1}^{k-1} L_i$.

The area of the drawing is polynomial. We now compute the length of $h(C)$, i.e., of the hypotenuse of an isosceles right triangle that contains the whole drawing, that has r as midpoint of its hypotenuse, and that has the hypotenuse forming angles of 120, 60, and 180 with edge (v_1, v_2). In the following refer to Fig. 5. Notice that the length of the longest edge of the drawing is at most equal to $h(C)$, while the length of the shortest edge of the drawing is at least 1, by construction.

We first notice that the drawing of P (without the drawing of subtrees $T_i's$) is contained inside an equilateral triangle Δ_e that has r as a vertex and such that the two sides incident to r have length equal $2c \sum_{i=1}^{k-1} L_i$ and form angles of 60 degrees with $h(C)$. In fact, the length of P is at most $2c \sum_{i=1}^{k-1} L_i$, and, since each edge of P forms an angle of 30 degrees with a horizontal line, the horizontal extension of P is at most $2c \sum_{i=1}^{k-1} L_i \cdot cos(30)$.

Consider the smallest isosceles right triangle Δ^* that contains Δ_e completely, that has r as midpoint of its hypotenuse, and that has the hypotenuse forming angles of 120, 60, and 180 with edge (v_1, v_2). Easy trigonometric calculations show that the hypotenuse of Δ^* has length at most $2(\cos(60) + \sin(60))(2c \sum_{i=1}^{k-1} L_i) = 5.46411c \sum_{i=1}^{k-1} L_i$.

Since edge (v_i, t_i) has length at most $L_i/(2 \cdot \tan(15)) \leq 1.867L_i$ and since all points of Δ_i are at distance at most $L_i/2$ from t_i, then no point of Δ_i is at distance greater than $2.367L_i$ from v_i. Consider the smallest isosceles right triangle Δ that contains Δ^*, that has r as midpoint of its hypotenuse, that has the hypotenuse forming angles of 120, 60, and 180 degrees with edge

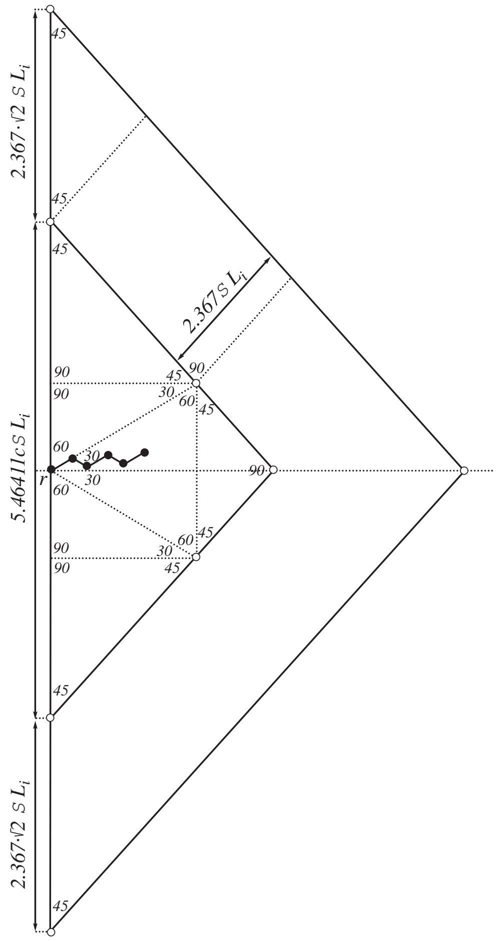

Figure 5: Bounding the constructed drawing with an isosceles right triangle.

(v_1, v_2), and such that every point on one of its catheti has distance at least $2.367\sum_{i=1}^{k-1} L_i$ from any point of Δ^*. It is easy to see that Δ contains the whole drawing, namely it contains P since it contains Δ^*, and it contains each subtree T_i, since T_i can stick outside Δ^* by at most $L_i/2 + 1.867L_i = 2.367L_i \leq 2.367\sum_{i=1}^{k-1} L_i$. Notice that the hypotenuse of Δ has length at most $5.46411c\sum_{i=1}^{k-1} L_i + 2(2.367\sqrt{2}\sum_{i=1}^{k-1} L_i)$. By choosing $c = 1.932$, the drawing of T is an MST embedding, and the length of the hypotenuse of smallest right isosceles triangle containing the drawing is bounded by $5.46411 \cdot 1.932\sum_{i=1}^{k-1} L_i + 2(2.367\sqrt{2}\sum_{i=1}^{k-1}) = 17.246\sum_{i=1}^{k-1} L_i$.

Lemma 1 *The length of* $h(C)$ *is at most* $17.246\sum_{i=1}^{k-1} L_i$*.*

Let $\alpha = 17.246$. Now, we express $h(C)$ as a function of the number of nodes of the tree. Denoting by $h(n)$ the maximum length of $h(C)$ when the input tree has n nodes, we inductively prove that $h(n) \leq n^{\log_2(3\alpha)}$. By Lemma 1, we get $h(n) \leq \alpha\sum_{i=1}^{k-1} h(n_i)$, where n_i is the number of nodes in T_i. By inductive hypothesis we get $h(n) \leq \alpha\sum_{i=1}^{k-1} n_i^{\log_2(3\alpha)}$. Group the numbers n_i in at most three groups N_1, N_2, and N_3 such that $\sum_{n_i \in N_1} n_i \leq \frac{n}{2}$, $\sum_{n_i \in N_2} n_i \leq \frac{n}{2}$, and $\sum_{n_i \in N_3} n_i \leq \frac{n}{2}$. Notice that it is always possible to construct such groups, namely start from groups $\{n_i\}$, each one containing a single value n_i, for $1 \leq i \leq k-1$. Since each subtree T_i has at most $n/2$ vertices, then $n_i \leq n/2$ and each starting group contains numbers adding up to at most $n/2$. Till there are more than three groups of numbers, consider any four groups of numbers. The numbers in the two groups that have minimal sum of their numbers add up to at most $n/2$ (otherwise the sum of the n_i's would be more than n). Hence, such groups can be joined to be the same group, hence decreasing the number of groups by one. Therefore, we have:

$$
\begin{aligned}
h(n) &\leq \alpha \sum_{i=1}^{k-1} n_i^{\log_2(3\alpha)} = \\
&= \alpha \left(\sum_{n_i \in N_1} n_i^{\log_2(3\alpha)} + \sum_{n_i \in N_2} n_i^{\log_2(3\alpha)} + \sum_{n_i \in N_3} n_i^{\log_2(3\alpha)} \right) \leq \\
&\leq \alpha \left(\left(\sum_{n_i \in N_1} n_i \right)^{\log_2(3\alpha)} + \left(\sum_{n_i \in N_2} n_i \right)^{\log_2(3\alpha)} + \left(\sum_{n_i \in N_3} n_i \right)^{\log_2(3\alpha)} \right) \leq \\
&\leq \alpha \left(\left(\frac{n}{2}\right)^{\log_2(3\alpha)} + \left(\frac{n}{2}\right)^{\log_2(3\alpha)} + \left(\frac{n}{2}\right)^{\log_2(3\alpha)} \right) \leq 3\alpha \left(\frac{n}{2}\right)^{\log_2(3\alpha)} = \\
&= 3\alpha \frac{n^{\log_2(3\alpha)}}{2^{\log_2(3\alpha)}} = 3\alpha \frac{n^{\log_2(3\alpha)}}{3\alpha} = n^{\log_2(3\alpha)},
\end{aligned}
$$

in which we used $\sum(n_i^k) \leq (\sum n_i)^k$. Hence, the inductive hypothesis is verified, and we can conclude that $h(n) \leq n^{\log_2 51.738} = O(n^{5.6932})$.

Finally, since the area of the drawing is the square of the length of its side, we get the following:

Theorem 2 *Every binary tree with n vertices admits an MST drawing in $O(n^{11.387})$ area.*

5 MST Embeddings of Complete Ternary Trees

In this section we deal with the construction of MST embeddings of complete ternary trees.

Let T be a complete ternary tree with n nodes and let $n = \frac{3^k-1}{2}$, for some integer k. Tree T consists of a root r and of three subtrees T_1, T_2, and T_3 rooted at the children r_1, r_2, and r_3 of r, respectively. Each of T_1, T_2, and T_3 has size less than $n/3$. We recursively embed T_1, T_2, and T_3 into three equal isosceles right triangles Δ_1, Δ_2, and Δ_3, respectively, so that the root of a subtree T_i is placed at the midpoint of the hypotenuse of Δ_i. In the base case, i.e., when T has only one node r, assume that r is placed at the midpoint of the hypotenuse of an isosceles right triangle Δ having the hypotenuse of length 1.

In the inductive case we construct a drawing of T inside an isosceles right triangle Δ as follows. Refer to Fig. 6. Let L denote the length of the hypotenuse of Δ_1, Δ_2, and Δ_3. Denote also by $a(\Delta_i)$, $b(\Delta_i)$, and $c(\Delta_i)$ the vertices of Δ_i, for $i = 1, 2, 3$, so that $a(\Delta_i)$ and $b(\Delta_i)$ are the end-vertices of the hypotenuse of Δ_i. Place r in the plane. Δ_2 is placed with its hypotenuse lying on a horizontal line, so that the segment connecting r and r_2 is perpendicular to the line through $a(\Delta_2)$ and $b(\Delta_2)$, and so that angles $r_2\hat{r}a(\Delta_2)$ and $r_2\hat{r}b(\Delta_2)$ are both of 10 degrees. Denote by d the distance between r and $a(\Delta_2)$. Δ_1 is placed with $a(\Delta_1)$ on the horizontal line through r, with $b(\Delta_1)$ at distance d from both r and $a(\Delta_2)$, so that angles $r_1\hat{r}a(\Delta_1)$ and $r_1\hat{r}b(\Delta_1)$ are both of 10 degrees, and so that segment $\overline{rr_1}$ is perpendicular to the line through $a(\Delta_1)$ and $b(\Delta_1)$. Δ_3 is placed in the plane symmetrically to Δ_1 with respect to a vertical line through r. The whole drawing is contained inside an isosceles right triangle Δ with hypotenuse that lies on a horizontal line and that has a length to be computed later.

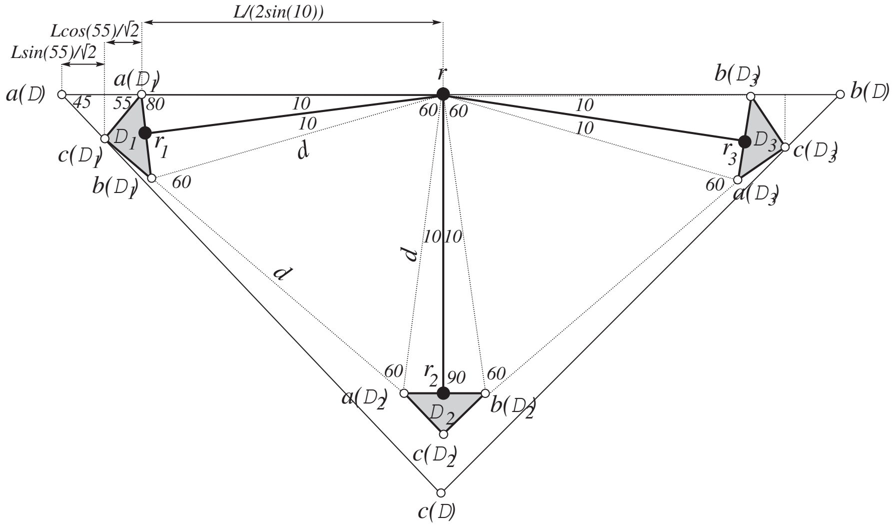

Figure 6: Bounding the constructed drawing with an isosceles right triangle.

We prove that the constructed drawing is an MST embedding of T. Inductively assume that the drawings of subtrees T_1, T_2, and T_3 are MST embeddings. We prove that each straight-line segment connecting a node w_1 in T_1 and a node w_2 in T_2 is longer than each edge of the path connecting w_1 and w_2 in T. By construction, the distance between w_1 and w_2 is at least d. The edges belonging to the path connecting w_1 and w_2 in T have length that is at most $\max\{L, L/(2 \cdot tan(10))\} = \max\{L, 2.836L\} = 2.836L$, namely all such edges are contained inside Δ_1 and Δ_2, but for (r, r_1) and (r, r_2), that by construction have length at most $L/(2 \cdot tan(10))$. Observe that, by construction, $d = L/(2 \cdot sin(10)) > 2.879L$. Hence, the distance between each pair of nodes w_1 and w_2 in T_1 and in T_2, respectively, satisfies the MST condition. It can be proved analogously that each pair of nodes w_2 and w_3 in T_2 and in T_3, respectively, satisfies the MST condition. Further, the MST condition is trivially satisfied for each pair of nodes w_1 and w_3 in T_1 and in T_3, respectively.

We now compute the area of the constructed drawing. Namely, we bound the constructed drawing by an isosceles right triangle Δ such that r is placed at the midpoint of the hypotenuse of Δ. Consider the line $l(\Delta_1)$ with slope -45 degrees passing through $c(\Delta_1)$. We claim that all the drawing is contained in the half-plane to the right of $l(\Delta_1)$. The claim is proved by the following two considerations: 1) Δ_1 is contained in the half-plane to the right of $l(\Delta_1)$, namely the slope of the segment connecting $c(\Delta_1)$ and $b(\Delta_1)$ is -35 degrees; 2) Δ_2 is contained in the half-plane to the right of $l(\Delta_1)$, namely the distance between r and $c(\Delta_2)$ is easily computed to be $L/(2 \cdot \tan(10)) + L/2 < 3.34L$, which is less than the distance between r and the intersection point of $l(\Delta_1)$ and the horizontal line through r. In fact, such a distance is equal to $\frac{L}{2 \cdot \sin(10)} + \frac{L \cdot \cos(55)}{\sqrt{2}} + \frac{L \cdot \sin(55)}{\sqrt{2}} > 3.864L$.

The length of the hypotenuse of Δ is twice the length of segment $\overline{ra(\Delta_1)}$, hence the hypotenuse of Δ has length less or equal than $7.7284L$. Observe that each edge of the drawing has length at least 1. Denote by $h(n)$ the length of the hypotenuse of Δ. We get: $h(n) \leq 7.7284h(\frac{n}{3}) \leq 7.7284^{\log_3 n} = n^{\log_3 7.7284} \leq n^{1.862}$. Since the area of Δ is asymptotically the square of its side, we obtain the following:

Theorem 3 *A complete ternary tree with n vertices admits an MST embedding in $O(n^{3.73})$ area.*

6 MST Embeddings of Arbitrary Ternary Trees

Now we present an algorithm to construct MST-embeddings of arbitrary ternary trees.

Overall strategy. Assume that the input ternary tree T is rooted at any node r of degree at most three. Analogously to the arbitrary binary tree case, select a chain $P = (r = v_1, v_2, v_3, \cdots, v_k)$ in T. Remove the chain from the tree, disconnecting the tree into several subtrees. Recursively draw the disconnected subtrees and place a drawing of the chain together with the drawings of the subtrees, obtaining a drawing of the whole tree.

Choice of the chain. The choice of P is done as in the arbitrary binary trees case. The first node v_1 of P is r. For each $1 \leq i < k$, node v_{i+1} is defined as the root of the larger of the three subtrees of v_i. Observe that each subtree of P has at most $n/2$ nodes.

The shape of the subtrees. Denote by T_i^1 and T_i^2 the subtrees rooted at the children t_i^1 and t_i^2 of v_i that do not belong to P, respectively. We recursively draw subtrees T_i^1 and T_i^2, for all $1 \leq i \leq k-1$, inside isosceles right triangles Δ_i^1 and Δ_i^2, respectively. For each $1 \leq i \leq k-1$, we scale up the drawing of the smallest between Δ_i^1 and Δ_i^2, so that the two isosceles right triangles are congruent. The whole chain together with the drawing of the subtrees of the nodes of P will be placed inside a larger isosceles right triangle Δ. The root of each subtree T_i^1 and T_2^1 is placed on the midpoint of the hypotenuse of Δ_i^1 and Δ_i^2, respectively. Denote by L_i the length of the hypotenuse of Δ_i^1 and Δ_i^2.

Drawing the chain and the subtrees together. Let $e_i = (v_i, v_{i+1})$, for $1 \leq i < k$. We draw P in a *zig-zag* way, with constant angles of 110 degrees between two consecutive edges e_i and e_{i+1}. See Fig. 7. The length of edges e_i will be determined later.

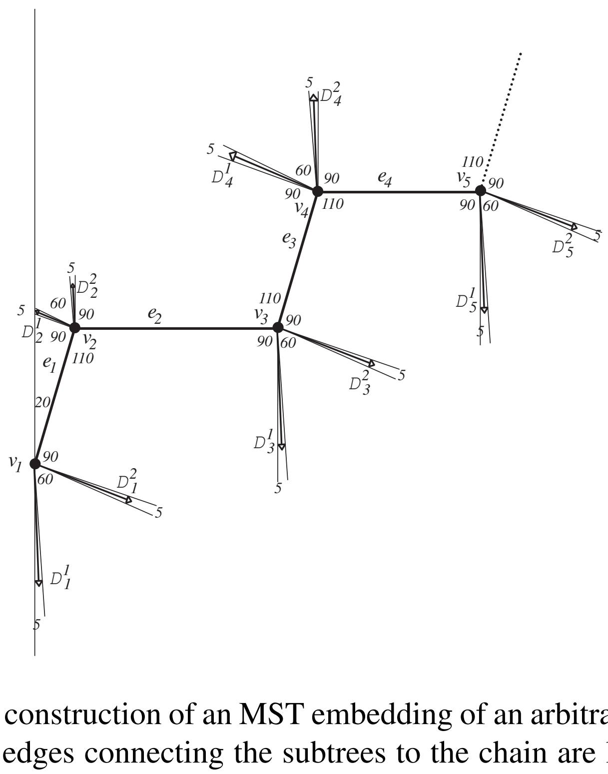

Figure 7: The recursive construction of an MST embedding of an arbitrary ternary tree. In order to improve readability, edges connecting the subtrees to the chain are longer than they should be (hence the actual drawing is not an MST embedding).

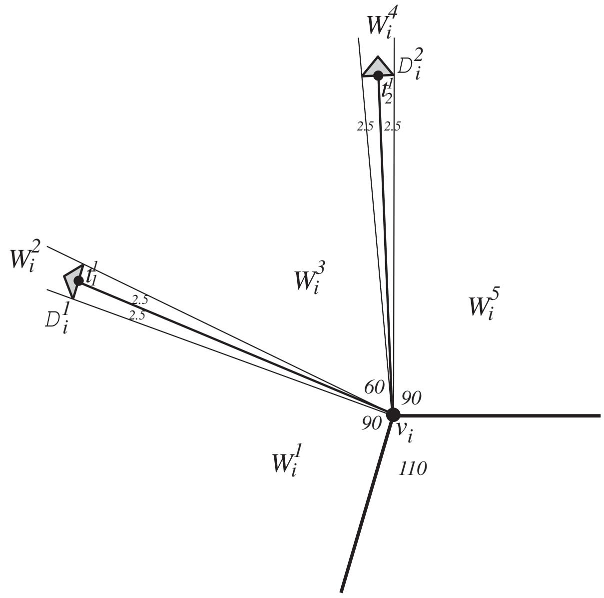

Figure 8: A closer look to the construction of an MST embedding of an arbitrary ternary tree.

Consider vertex v_i. Opposite to the 110 degree angle, we have an angle of 250 degrees, which we partition into five consecutive wedges W_i^1, W_i^2, W_i^3, W_i^4, and W_i^5 of 90, 5, 60, 5, and 90 degrees, respectively, such that W_i^1 is the wedge closer to vertex v_{i-1}. We place Δ_i^1 inside W_i^2 and Δ_i^2 inside W_i^4 as follows. Consider the line l_i^2 through v_i bisecting W_i^2. Vertex t_i^1 is placed on l_i^2 and triangle Δ_i^1 is placed inside W_i^2 so that the hypotenuse of Δ_i^1 is perpendicular to l_i^2, and so that the endvertices of the hypotenuse of Δ_i^1 lie on the semi-axes delimiting W_i^2. Δ_i^2 is analogously placed inside W_i^4. See Fig. 8.

Notice that, for vertex v_1 (and for vertex v_k), wedges W_i^1, W_i^2, W_i^3, W_i^4, and W_i^5 are not well-defined, since only one edge e_1 of P is incident to v_1. However, it is not difficult to extend the above definition of wedges W_i^1, W_i^2, W_i^3, W_i^4, and W_i^5 to the case in which $i = 1$, by considering a dummy edge (v_0, v_1) that has an angle of 110 degrees with edge (v_1, v_2), and defining the wedges incident to v_1 as for the other vertices of P.

Choosing the length of edges $\mathbf{e_i}$. As in the arbitrary binary tree case, we set:

$$len(e_i) = max\{cL_i, cL_{i+1}\},$$

where c is a constant to be determined later. In order to have length at least one for all edges, we set $len(e_i) = 1$, for all edges e_i where none of subtrees T_i^1, T_i^2, T_{i+1}^1, and T_{i+1}^2 exists.

The isosceles right triangle Δ is defined as the smallest isosceles right triangle containing the whole drawing, having r as midpoint of the hypotenuse, and having the hypotenuse forming angles of 160, 20, and 180 with edge (v_1, v_2). In the following we suppose, for clarity of exposition, that the hypotenuse of Δ is vertical, and that P is contained in the half-plane to the right of the line through the hypotenuse. If a tree T has only one node, Δ is defined as the isosceles right triangle having r as midpoint of the hypotenuse, and having the hypotenuse of length 1.

The drawing satisfies the MST condition. We use induction to show that every pair of vertices in the drawing satisfies the MST condition. If the tree has only one node, then there is nothing to prove. Otherwise, inductively suppose that each pair of nodes in the drawing of each subtree T_i^1 and T_i^2 satisfies the MST condition. Then, we prove that each pair of nodes in the whole drawing satisfies the MST condition.

The only pairs of nodes for which the MST condition is not trivially satisfied, are: (i) node v_i and any node in T_i^1 or in T_i^2, for $i = 1, 2, \cdots, k-1$, (ii) any node in T_i^1 and any node in T_i^2, for $i = 1, 2, \cdots, k-1$, (iii) node v_i and any node in T_{i-1}^2, for $i = 2, 3, \cdots, k$, (iv) node v_i and any node in T_{i+1}^1, for $i = 1, 2, \cdots, k-2$, and (v) any node of $T_{i-1}^2 \cup \{v_{i-1}\}$ and any node in $T_{i+1}^1 \cup \{v_{i+1}\}$, for $i = 2, 3, \cdots, k-2$.

(i) Consider node v_i and any node w_i in T_i^1 (resp. in T_i^2), for any $i = 1, 2, \cdots, k-1$. We prove that all edges in the path from v_i to w_i are shorter than segment $\overline{v_i w_i}$. The length of each edge of such a path belonging to T_i^1 (resp. to T_i^2) is at most L_i. The length of edge (v_i, t_i^1) (resp. edge (v_i, t_i^2)) is equal to $L_i/(2 \cdot \tan(2.5)) \geq 11.451 L_i$. Hence, (v_i, t_i^1) (resp. (v_i, t_i^2)) is the longest edge of the path connecting v_i and w_i. However, segment $\overline{v_i w_i}$ is longer than (v_i, t_i^1) (resp. than (v_i, t_i^2)), since w_i is contained inside Δ_i^1 (resp. inside Δ_i^2), whose closest point to v_i is t_i^1 (resp. t_i^2).

(ii) Consider any node n_i^1 in T_i^1 and any node n_i^2 in T_i^2, for any $i = 1, 2, \cdots, k-1$. We prove that all edges in the path from n_i^1 to n_i^2 are shorter than segment $\overline{n_i^1 n_i^2}$. The length of each edge of such a path belonging also to T_i^1 or to T_i^2 is at most L_i. Further, edges (v_i, t_i^1) and (v_i, t_i^2) have length $L_i/(2 \cdot \tan(2.5)) \approx 11.451 L_i$. Hence, (v_i, t_i^1) and (v_i, t_i^2) are the longest edges in the path connecting n_i^1 and n_i^2. However, consider the intersection point $p(i_1)$ of Δ_i^1 and the line separating wedges W_i^2 and W_i^3, and consider the intersection point $p(i_2)$ of Δ_i^2 and the line separating wedges W_i^3 and W_i^4. The length of segment $\overline{n_i^1 n_i^2}$ is greater or equal than the length of segment $\overline{p(i_1) p(i_2)}$. By construction, triangle $(p(i_1), p(i_2), v_i)$ is equilateral, hence $\overline{p(i_1) p(i_2)}$ has the same length of segments $\overline{v_i p(i_1)}$ and $\overline{v_i p(i_2)}$, that is $L_i/(2 \cdot \sin(2.5)) \approx 11.462 L_i$, which is greater than $L_i/(2 \cdot \tan(2.5))$.

(iii) For any $i = 2, 3, \cdots, k-1$, consider a node v_i and any node w_{i-1} in T_{i-1}^2, and suppose that the pair (v_i, w_{i-1}) of vertices does not satisfy the MST condition. As in the previous case each edge of such a path belonging also to T_{i-1}^2 has length at most L_{i-1}. Further, edge (v_{i-1}, t_{i-1}^2) has length $L_{i_1}/(2 \cdot \tan(2.5)) \leq 11.452 L_{i_1}$, and edge (v_{i-1}, v_i) has length at least cL_{i-1}. It follows that, as long as $c \geq 11.452$, edge (v_{i-1}, v_i) is the longest edge in the path connecting v_i and w_{i-1}. However, consider triangle (v_i, v_{i-1}, w_{i-1}). By construction, angle $v_i \widehat{v}_{i-1} w_{i-1}$ contains wedge W_i^5 and hence it is greater or equal than 90 degrees. Segment $\overline{v_i w_{i-1}}$ is opposite to $v_i \widehat{v}_{i-1} w_{i-1}$ and hence is the longest side of such a triangle. It follows that $\overline{v_i w_i}$ is longer than (v_{i-1}, v_i).

(iv) For any $i = 1, 2, \cdots, k-2$, it can be proved analogously to the previous case that the MST of the points of the drawing cannot contain an edge (v_i, w_{i+1}), for any node w_{i+1} in T_{i+1}^1.

(v) Consider any node w_{i-1} in $T_{i-1}^2 \cup \{v_{i-1}\}$ and any node w_{i+1} in $T_{i+1}^1 \cup \{v_{i+1}\}$, for $i = 2, 3, \cdots, k-2$. The path P_{i-1}^{i+1} connecting w_{i-1} and w_{i+1} in T contains edges e_{i-1} and e_i. All edges of P_{i-1}^{i+1} belonging to T_{i-1}^2 or to T_{i+1}^1 are contained inside Δ_{i-1}^2 or Δ_{i+1}^1, respectively, and hence their length is at most the maximum between L_{i-1} and L_{i+1}. Further, the length of edge (v_{i-1}, t_{i-1}^2) is $L_{i-1}/(2 \cdot \tan(2.5)) \leq 11.452 L_{i-1}$. Analogously, the length of edge (v_{i+1}, t_{i+1}^1) is at most $11.452 L_{i+1}$. Hence, the length of each edge in P_{i-1}^{i+1} is less or equal than $\max\{11.452 L_{i-1}, 11.452 L_{i+1}, len(e_{i-1}), len(e_i)\}$. Observe that, by construction, $len(e_{i-1}) \geq cL_{i-1}$, and that $len(e_i) \geq cL_{i+1}$. Hence, as long as $c \geq 11.452$, one between e_{i-1} and e_i is the longest edge in P_{i-1}^{i+1}, and we have only to prove that the distance between w_{i-1} and w_{i+1} is greater than $\max\{len(e_{i-1}), len(e_i)\}$. In the following, refer to Figs. 9 and 10.

Consider line $l_{i-1}^{4,5}$ separating wedges W_{i-1}^4 and W_{i-1}^5, and consider line $l_i^{1,2}$ separating wedges W_i^1 and W_i^2. By construction such lines are parallel. Further, T_{i-1}^2 is contained in

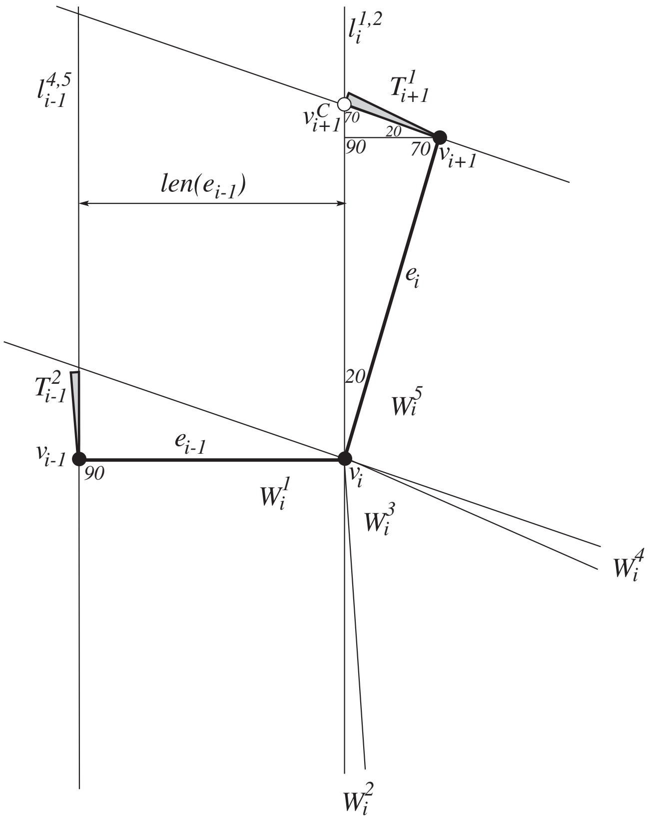

Figure 9: Illustration for the proof that the MST condition is satisfied for any node of $T^2_{i-1} \cup \{v_{i-1}\}$ and any node in $T^1_{i+1} \cup \{v_{i+1}\}$, for $i = 2, 3, \cdots, k-2$.

the half-plane delimited by $l^{4,5}_{i-1}$ and not containing $l^{1,2}_i$. Notice that the distance between $l^{4,5}_{i-1}$ and $l^{1,2}_i$ is exactly $len(e_{i-1})$. We claim that, for a suitable constant c, T^1_{i+1} is entirely contained in the half-plane delimited by $l^{1,2}_i$ and not containing $l^{4,5}_{i-1}$. The claim clearly implies that the distance between w_{i-1} and w_{i+1} is greater or equal than $len(e_{i-1})$.

Since the length of edge (v_{i+1}, t^1_{i+1}) is $L_{i+1}/(2 \cdot \tan(2.5)) \leq 11.452 L_{i+1}$ and since all points of Δ^1_{i+1} are at distance at most $L_{i+1}/2$ from t^1_{i+1}, then no point of Δ^1_{i+1} is at distance greater than $11.952 L_{i+1}$ from v_{i+1}. Hence, Δ^1_{i+1} is enclosed inside an isosceles triangle $\overline{\Delta}$ having v_{i+1} as a vertex incident to a 5-degree angle, and having two sides (v_{i+1}, v^C_{i+1}) and (v_{i+1}, v^D_{i+1}) of length $11.952 L_{i+1}/cos(2.5) \leq 11.964 L_{i+1}$ lying on the line $l^{1,2}_{i+1}$ separating wedges W^1_{i+1} and W^2_{i+1}, and on the line $l^{2,3}_{i+1}$ separating wedges W^2_{i+1} and W^3_{i+1}, respectively.

We show that, for a suitable value of c, $\overline{\Delta}$ is entirely contained in the half-plane delimited by $l^{1,2}_i$ and not containing $l^{4,5}_{i-1}$. First, observe that v^C_{i+1} is the point of $\overline{\Delta}$ closer to $l^{1,2}_i$. The distance between v_{i+1} and $l^{1,2}_i$ is equal to $len(e_i) \cdot \sin(20) \geq 0.342 c L_{i+1}$. The distance between v^C_{i+1} and v_{i+1} in the direction orthogonal to $l^{1,2}_i$ is $11.964 L_{i+1} \cdot \cos(20) \leq 11.243 L_{i+1}$. It follows that, as long as $0.342 c L_{i+1} \geq 11.243 L_{i+1}$, i.e., as long as $c \geq 32.875$, v^C_{i+1} (and hence $\overline{\Delta}$ and Δ^1_{i+1}) is in the half-plane delimited by $l^{1,2}_i$ and not containing $l^{4,5}_{i-1}$.

In analogous way, it can be proved that, as long as $c \geq 32.875$, the distance between w_{i-1} and w_{i+1} is greater than $len(e_i)$. Hence, as long as $c \geq 32.875$, the straight-line segment between w_{i-1} and w_{i+1} is longer than every edge in the path P^{i+1}_{i-1} connecting w_{i-1} and w_{i+1} in T, and hence it does not belong to the MST of the points of the drawing.

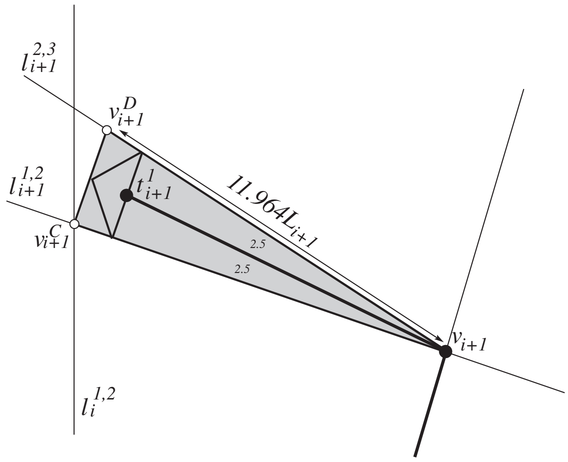

Figure 10: Triangle $\overline{\Delta}$, shaded in the picture, containing edge (v_{i+1}, t_{i+1}) and subtree T_{i+1}^1.

The length of P. We bound the length of P as a function of the lengths L_i's. As in the binary case, since $len(e_i) = max\{cL_i, cL_{i+1}\}$, and since $len(e_i) \geq 1$, for every $1 \leq i < k$, then $len(e_i) < cL_i + cL_{i+1}$. It follows that $\sum_{i=1}^{k-1} len(e_i) \leq 2c \sum_{i=1}^{k-1} L_i$.

The area of the drawing is polynomial. We now compute the length of $h(C)$, i.e., of the hypotenuse of an isosceles right triangle that contains the whole drawing, that has r as midpoint of its hypotenuse, and that has the hypotenuse forming angles of 160, 20, and 180 degrees with edge (v_1, v_2). In the following refer to Fig. 11. Notice that the length of the longest edge of the drawing is at most equal to $h(C)$, while the length of the shortest edge of the drawing is at least 1, by construction.

The computation of the area of the drawing proceeds as in the binary case. We first notice that the drawing of P (without the drawing of subtrees T_i^1's and T_i^2's) is contained inside an isosceles triangle Δ_e such that:

- Δ_e has two angles of 20 degrees and one angle of 140 degrees;
- r is the vertex of Δ_e incident to the 140-degree angle;
- one side of Δ_e contains edge (v_1, v_2);
- the distance between r and the side of Δ_e opposite to r is $2c \sum_{i=1}^{k-1} L_i$.

In fact, the length of P is at most $2c \sum_{i=1}^{k-1} L_i$, and, no edge of P forms an angle of less than 20 degrees with a vertical line.

Consider the smallest isosceles right triangle Δ^* that contains Δ_e completely, that has r as midpoint of its hypotenuse, and that has the hypotenuse forming angles of 160, 20, and 180 degrees with edge (v_1, v_2). Easy trigonometric calculations show that the hypotenuse of Δ^* has length at most $2(1 + 1/\tan(20))(2c \sum_{i=1}^{k-1} L_i) < 14.99c \sum_{i=1}^{k-1} L_i$.

Consider the smallest isosceles right triangle Δ that contains Δ^*, that has r as midpoint of its hypotenuse, that has the hypotenuse forming angles of 160, 20, and 180 degrees with edge (v_1, v_2), and such that every point on one of its catheti has distance at least $11.952 \sum_{i=1}^{k-1} L_i$ from any point of Δ^*. It is easy to see that Δ contains the whole drawing, namely it contains P since it contains Δ^*, and it contains each subtree T_i^1 and T_i^2, since T_i^1 and T_i^2 can stick outside Δ^* by at most $L_i/2 + 11.452L_i = 11.952L_i \leq 11.952 \sum_{i=1}^{k-1} L_i$. Notice that the hypotenuse of Δ has

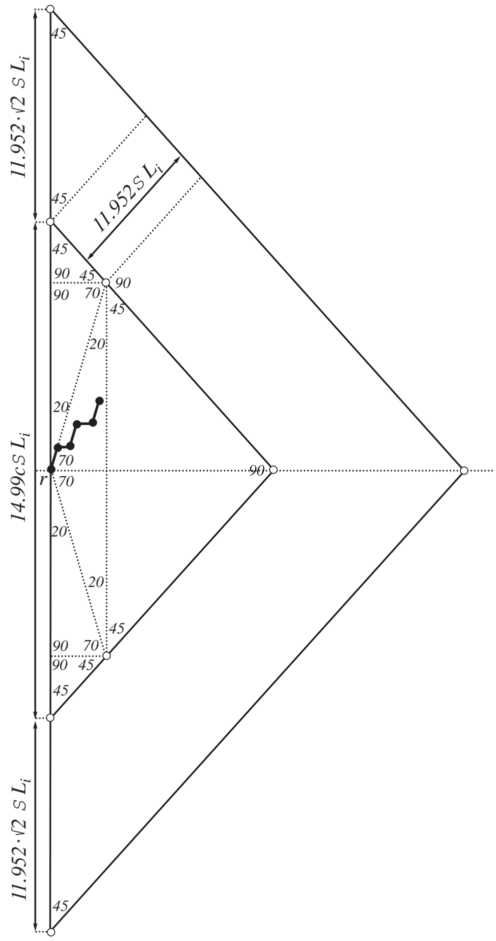

Figure 11: Bounding the constructed drawing with an isosceles right triangle.

length at most $14.99c\sum_{i=1}^{k-1} L_i + 2(11.952\sqrt{2}\sum_{i=1}^{k-1} L_i)$. By choosing $c = 32.875$, the drawing of T is an MST embedding, and the length of $h(C)$ is bounded by $14.99 \cdot 32.875\sum_{i=1}^{k-1} L_i + 2(11.952\sqrt{2}\sum_{i=1}^{k-1} L_i) < 526.602\sum_{i=1}^{k-1} L_i$.

Lemma 2 *The length of $h(C)$ is at most $526.602\sum_{i=1}^{k-1} L_i$.*

Let $\alpha = 526.602$. We express $h(C)$ as a function of the number of nodes of the tree. Denote by $h(n)$ the maximum length of $h(C)$ when the input tree has n nodes. It can be inductively proved that $h(n) \leq n^{\log_2(3\alpha)}$. However, this is done by using *exactly* the same arguments and calculations that we used for the binary case, and hence such arguments and calculations are omitted here. Then, we conclude that $h(n) \leq n^{\log_2 1579.805} = O(n^{10.626})$.

Finally, since the area of the drawing is the square of the length of its side, we get the following:

Theorem 4 *Every ternary tree with n vertices admits an MST drawing in $O(n^{21.252})$ area.*

7 Discussion and Conclusion

In this paper, we have shown algorithms for constructing MST embeddings of trees with maximum degree 4 in polynomial area. It would be interesting to understand how much the bounds achieved by our algorithms can be improved by modifying the constant angles in the geometric constructions we have shown. In the case of complete binary trees, a construction similar to the one we presented for complete ternary trees achieve a slightly better bound than the one claimed in Theorem 1 (we still opted for providing the construction of Section 3, which is particularly simple).

It is an obvious and important open problem to determine whether polynomial-area suffices for constructing MST embeddings of trees with degree 5 or, instead, the conjecture of Monma and Suri is correct [11]. We remark that such an exponential area lower bound can be quite easily obtained if the order of the edges incident to the nodes of the tree is fixed. In order to prove such a lower bound, it is sufficient to consider a 5-regular caterpillar, i.e. a 5-regular tree T such that removing all the leaves turns T into a path called *backbone*, in which starting at the first edge (v_1, v_2) of the backbone $v_1, v_2, \cdots, v_k$, the next edge (v_i, v_{i+1}) of the backbone can be found to be the next edge to the right of the current backbone edge, for each $i = 1, 2, \ldots, k-1$ (see Fig. 12).

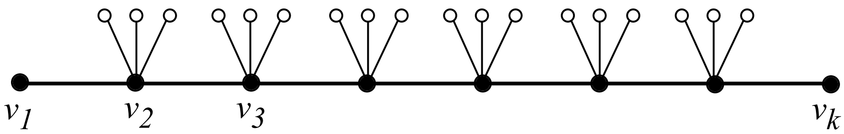

Figure 12: A 5-regular caterpillar in which the backbone always turns right.

However, we expect that there exist degree-5 trees for which exponential area is required in any MST embedding, even if the order of the edges incident to each node is not fixed. In fact, we have the following conjecture:

Conjecture 1 *The* 5*-regular caterpillar requires exponential area.*

Acknowledgements:

This work has been initiated at the Workshop on Graph Drawing and Computational Geometry in Bertinoro, Italy. Thanks to the organizers and participants for the inspiring atmosphere. The initial results on complete binary trees were achieved together with Roberto Tamassia. Special thanks also go to Markus Geyer and Barbara Pampel for many critical remarks und suggestions.

References

[1] S. Arora. Polynomial time approximation schemes for euclidean traveling salesman and other geometric problems. *J. ACM*, 45(5):753–782, 1998.

[2] S. Arora and K. L. Chang. Approximation schemes for degree-restricted mst and red-blue separation problems. *Algorithmica*, 40(3):189–210, 2004.

[3] T. M. Chan. Euclidean bounded-degree spanning tree ratios. *Discrete & Computational Geometry*, 32(2):177–194, 2004.

[4] H. Cheng, Q. Liu, and X. Jia. Heuristic algorithms for real-time data aggregation in wireless sensor networks. In S. Onoe, M. Guizani, H.-H. Chen, and M. Sawahashi, editors, *IWCMC*, pages 1123–1128, 2006.

[5] P. Eades and S. Whitesides. The realization problem for euclidean minimum spanning trees in np-hard. *Algorithmica*, 16(1):60–82, 1996.

[6] M. R. Garey and D. S. Johnson. *Computers and Intractability: A Guide to the Theory of NP-Completeness*. W. H. Freeman, 1979.

[7] J. A. King. Realization of degree 10 minimum spanning trees in 3-space. In *CCCG*, 2006.

[8] J. Leech. The problem of the thirteen spheres. *The Mathematical Gazette*, 40:22–23, 1956.

[9] G. Liotta and G. Di Battista. Computing proximity drawings of trees in the 3-dimensional space. In S. G. Akl, F. K. H. A. Dehne, J.-R. Sack, and N. Santoro, editors, *Workshop on Algorithms and Data Structures*, pages 239–250, 1995.

[10] J. S. B. Mitchell. Guillotine subdivisions approximate polygonal subdivisions: A simple polynomial-time approximation scheme for geometric tsp, k-mst, and related problems. *SIAM J. Comput.*, 28(4):1298–1309, 1999.

[11] C. L. Monma and S. Suri. Transitions in geometric minimum spanning trees. *Discrete & Computational Geometry*, 8:265–293, 1992.

[12] C. H. Papadimitriou and U. V. Vazirani. On two geometric problems related to the traveling salesman problem. *J. Algorithms*, 5:231–246, 1984.